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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 3 (WFM03)

Further Pure 3 Practice Paper 3

Worked Solutions

AIMING FOR A/A*

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM03/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	Hyperbolic expressions in the form $R\sinh(x+\alpha)$	10	2
2	An inverse hyperbolic function and its domain	10	3
3	Diagonalising with a repeated eigenvalue	11	4
4	A reflection in a plane and a parallel plane	11	5
5	A reduction formula for $\tanh^n x$	11	6
6	A tangent to an ellipse, an area and a locus	11	7
7	Arc length and surface area by substitution	11	8

Question paper & mark scheme | Exam questions by topic | More practice papers

1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(x) = 3\cosh x + 5\sinh x$$

(a) Explain why $f(x)$ cannot be written in the form $R\cosh(x+\alpha)$, where $R > 0$, and express $f(x)$ in the form $R\sinh(x+\alpha)$, where $R > 0$ and α is an exact natural logarithm.

(4)

(b) Hence, or otherwise, solve the equation $f(x) = 4$, giving your answer as a single natural logarithm.

(3)

(c) Solve the equation

$$5\cosh x + 3\sinh x = 4$$

and explain why it has exactly one solution.

(3)

(a) $R\cosh(x+\alpha) = R\cosh\alpha\cosh x + R\sinh\alpha\sinh x$ needs $R\cosh\alpha = 3$, $R\sinh\alpha = 5$

$\Rightarrow \tanh\alpha = \frac{5}{3} > 1$, impossible since $|\tanh\alpha| < 1$ B1

$R\sinh(x+\alpha) = R\cosh\alpha\sinh x + R\sinh\alpha\cosh x$: $R\cosh\alpha = 5$, $R\sinh\alpha = 3$ M1

$\cosh^2\alpha - \sinh^2\alpha = 1$: $R^2 = 25 - 9 = 16$, $R = 4$ A1

$\tanh\alpha = \frac{3}{5}$: $\alpha = \frac{1}{2}\ln\frac{1+3/5}{1-3/5} = \frac{1}{2}\ln 4 = \ln 2$ $\boxed{f(x) = 4\sinh(x+\ln 2)}$ A1

(b) $4\sinh(x+\ln 2) = 4 \Rightarrow \sinh(x+\ln 2) = 1$ M1

$x + \ln 2 = \operatorname{arsinh} 1 = \ln(1+\sqrt{2})$ M1

$\boxed{x = \ln((1+\sqrt{2})/2)}$ (by exponentials, $e^x = (1-\sqrt{2})/2 < 0$ is rejected) A1

(c) $R\cosh\alpha = 5$, $R\sinh\alpha = 3$: same R, α, so $4\cosh(x+\ln 2) = 4$ M1

$\cosh(x+\ln 2) = 1 \Rightarrow x + \ln 2 = 0$, $\boxed{x = -\ln 2}$ A1

One solution: $\cosh u \geqslant 1$ and equals 1 only at $u = 0$ (or $(2e^x - 1)^2 = 0$, a repeated root) B1

(Total for Question 1 is 10 marks)

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2.

$$f(x) = \text{artanh}\left(\frac{x-1}{x+1}\right)$$

(a) Find the set of values of x for which $f(x)$ is defined.

(2)

(b) Find $f'(x)$ in its simplest form, and hence express $f(x)$ in terms of $\ln x$

(4)

(c) Use integration by parts to find the exact value of

$$\int_1^e x\,\text{artanh}\left(\frac{x-1}{x+1}\right)dx$$

(4)

(a) artanh is defined on $(-1, 1)$: $-1 < \frac{x-1}{x+1} < 1$ M1

$\frac{x-1}{x+1} < 1 \Rightarrow \frac{-2}{x+1} < 0 \Rightarrow x > -1$; $\frac{x-1}{x+1} > -1 \Rightarrow \frac{2x}{x+1} > 0 \Rightarrow x > 0$ or $x < -1$

$\boxed{x > 0}$ A1

(b) $f'(x) = \frac{1}{1-((x-1)/(x+1))^2} \times \frac{(x+1)-(x-1)}{(x+1)^2}$ M1

$= \frac{(x+1)^2}{(x+1)^2-(x-1)^2} \times \frac{2}{(x+1)^2} = \frac{2}{4x}$ A1

$\boxed{f'(x) = 1/(2x)}$ A1

$f(x) = \frac{1}{2}\ln x + c$; $f(1) = \text{artanh } 0 = 0 \Rightarrow c = 0$: $\boxed{f(x) = (1/2)\ln x}$, $x > 0$ B1

(c) $u = f(x)$, $dv/dx = x$: $[\frac{1}{2}x^2 f(x)] - \int \frac{1}{2}x^2 \times \frac{1}{2x}\,dx$ M1 A1

$= [\frac{1}{2}x^2 f(x) - \frac{1}{8}x^2]_1^e$, $f(e) = \frac{1}{2}\ln e = \frac{1}{2}$, $f(1) = 0$ M1

$= \frac{e^2}{4} - \frac{e^2}{8} - (0 - \frac{1}{8}) = \boxed{(e^2+1)/8}$ A1

(Total for Question 2 is 10 marks)

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3.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

(a) Show that $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue.

(2)

(b) Find the other eigenvalues of $\mathbf{A}$, showing that one of them is repeated.

(4)

(c) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$

(5)

(a) $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ M1

eigenvalue 5 A1

(b) $\det(A - \lambda I) = (3-\lambda)[(3-\lambda)^2 - 1] + [1 - (3-\lambda)] + [1 - (3-\lambda)]$ M1

Let $w = 3 - \lambda$: $w^3 - 3w + 2$ i.e. $-\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$ A1

$\lambda = 5$ is a root: $-(\lambda - 5)(\lambda^2 - 4\lambda + 4) = 0$ M1

$-(\lambda - 5)(\lambda - 2)^2 = 0$: $\lambda = 2$, repeated A1

(c) $\lambda = 2$: every row of $A - 2I$ gives $x - y + z = 0$ (a whole plane of eigenvectors) M1

Choose $(1, 1, 0)$ ✓ A1

The second must ALSO be orthogonal to $(1, 1, 0)$: $(1, 1, 0) \times (1, -1, 1)$ M1

$= (1 \times 1 - 0,\ 0 - 1 \times 1,\ -1 - 1) = (1, -1, -2)$ check $1 + 1 - 2 = 0$ ✓ A1

$P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$ $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ A1

(Total for Question 3 is 11 marks)

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4. The plane Π_1 has equation $2x - y + 2z = 3$ and the point A has coordinates $(5, -1, 5)$

(a) Find the coordinates of A', the reflection of A in Π_1

(5)

The plane Π_2 is parallel to Π_1, and the perpendicular distance between Π_1 and Π_2 is 2. Π_2 lies between A and Π_1

(b) Find a Cartesian equation of Π_2

(3)

(c) Find the shortest distance from A' to Π_2

(3)

(a) Line through A along $n = (2, -1, 2)$: $r = (5, -1, 5) + t(2, -1, 2)$ M1

Meets Π_1: $2(5 + 2t) - (-1 - t) + 2(5 + 2t) = 3 \Rightarrow 21 + 9t = 3$ M1

$t = -2$: foot of the perpendicular $(1, 1, 1)$ check $2 - 1 + 2 = 3$ ✓ A1

Reflection: twice as far, $t = -4$ M1

$\boxed{A'(-3, 3, -3)}$ A1

(b) Π_2: $2x - y + 2z = d$, distance $|d - 3|/\sqrt{4 + 1 + 4} = |d - 3|/3 = 2$ M1

$d = 9$ or $d = -3$ (two planes, one each side of Π_1) A1

At A: $10 + 1 + 10 = 21$, so "between" needs $3 < d < 21$: $\boxed{2x - y + 2z = 9}$ A1

(c) At A': $2(-3) - 3 + 2(-3) = -15$ M1

Distance $= \frac{|-15 - 9|}{3}$ M1

$= \boxed{8}$ (A' is 6 beyond Π_1, then 2 more to Π_2) A1

(Total for Question 4 is 11 marks)

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5.

$$I_n = \int_0^{\ln 3} \tanh^n x \, dx, \qquad n \geqslant 0$$

(a) Show that, for $n \geqslant 2$,

$$I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{4}{5}\right)^{n-1}$$

(5)

(b) Find the exact value of I_1

(2)

(c) Find the exact value of I_5

(3)

(d) Without evaluating any more integrals, explain why $I_6 < I_5$

(1)

(a) $\tanh^n x = \tanh^{n-2} x \times \tanh^2 x = \tanh^{n-2} x (1 - \operatorname{sech}^2 x)$ M1 A1

$I_n = I_{n-2} - \int_0^{\ln 3} \tanh^{n-2} x \operatorname{sech}^2 x \, dx$ M1

$= I_{n-2} - \left[\frac{\tanh^{n-1} x}{n-1}\right]_0^{\ln 3}$ (since $d/dx \tanh x = \operatorname{sech}^2 x$) A1

$\tanh(\ln 3) = \frac{3 - 1/3}{3 + 1/3} = \frac{4}{5}$, $\tanh 0 = 0$: $I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{4}{5}\right)^{n-1}$ as required A1

(b) $I_1 = \int_0^{\ln 3} \tanh x \, dx = [\ln \cosh x]_0^{\ln 3}$ M1

$\cosh(\ln 3) = \frac{3 + 1/3}{2} = \frac{5}{3}$, $\cosh 0 = 1$: $\boxed{I_1 = \ln(5/3)}$ A1

(c) $n = 3$: $I_3 = I_1 - \frac{1}{2}\left(\frac{4}{5}\right)^2 = \ln\frac{5}{3} - \frac{8}{25}$ M1

$n = 5$: $I_5 = I_3 - \frac{1}{4}\left(\frac{4}{5}\right)^4 = \ln\frac{5}{3} - \frac{8}{25} - \frac{64}{625}$ M1

$\boxed{I_5 = \ln(5/3) - 264/625}$ A1

(d) On $[0, \ln 3]$, $0 \leqslant \tanh x < 1$, so $\tanh^6 x \leqslant \tanh^5 x$ (equal only at $x = 0$) $\Rightarrow I_6 < I_5$ B1

(Total for Question 5 is 11 marks)

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6.

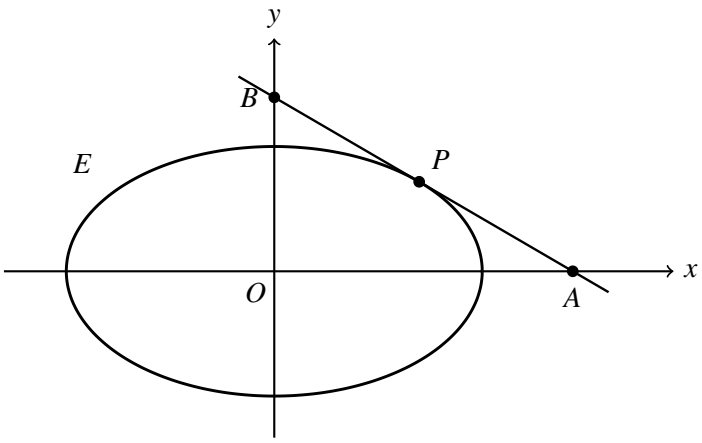

Figure 1

Figure 1 shows the ellipse E with equation $\frac{x^2}{25}+\frac{y^2}{9}=1$. The point $P(5\cos\theta, 3\sin\theta)$, $0<\theta<\frac{\pi}{2}$, lies on E. The tangent to E at P meets the x-axis at A and the y-axis at B

(a) Find an equation of the tangent to E at P, giving your answer in the form $px\cos\theta+qy\sin\theta=r$, where p, q and r are integers.

(4)

(b) Find the least possible area of triangle OAB, where O is the origin, and the exact coordinates of P when it occurs.

(4)

(c) The point M is the midpoint of AB. Find a Cartesian equation of the locus of M as θ varies, stating any restrictions on x and y

(3)

(a) $\frac{dx}{d\theta} = -5\sin\theta$, $\frac{dy}{d\theta} = 3\cos\theta \Rightarrow \frac{dy}{dx} = -\frac{3\cos\theta}{5\sin\theta}$ M1 A1

$y - 3\sin\theta = -\frac{3\cos\theta}{5\sin\theta}(x - 5\cos\theta)$ M1

$\times 5\sin\theta$: $5y\sin\theta - 15\sin^2\theta = -3x\cos\theta + 15\cos^2\theta \Rightarrow$ $3x\cos\theta + 5y\sin\theta = 15$ A1

(b) $y=0$: $x = \frac{5}{\cos\theta}$; $x=0$: $y = \frac{3}{\sin\theta}$ $A(5/\cos\theta, 0)$, $B(0, 3/\sin\theta)$ B1

Area $OAB = \frac{1}{2}\times\frac{5}{\cos\theta}\times\frac{3}{\sin\theta} = \frac{15}{2\sin\theta\cos\theta} = \frac{15}{\sin 2\theta}$ M1

Least when $\sin 2\theta = 1$, $\theta = \frac{\pi}{4}$; least area 15 $P(5\sqrt{2}/2, 3\sqrt{2}/2)$ A1 A1

(c) $M = \left(\frac{5}{2\cos\theta}, \frac{3}{2\sin\theta}\right) \Rightarrow \cos\theta = \frac{5}{2x}$, $\sin\theta = \frac{3}{2y}$ M1

$\cos^2\theta + \sin^2\theta = 1$: $\frac{25}{4x^2} + \frac{9}{4y^2} = 1$ M1

$9x^2 + 25y^2 = 4x^2y^2$, $x > 5/2$, $y > 3/2$ $(0 < \cos\theta, \sin\theta < 1)$ A1

(Total for Question 6 is 11 marks)

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7. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{1}{2}x^2, \qquad 0 \leqslant x \leqslant \sqrt{3}$$

(a) Use the substitution $x = \sinh u$ to find the exact length of C, giving your answer in the form $a\sqrt{3} + b\ln\left(2+\sqrt{3}\right)$, where a and b are rational numbers.

(5)

(b) The curve C is rotated through 2π radians about the x-axis. Find the exact area of the surface generated, giving your answer in the form $\pi\left(p\sqrt{3} - q\ln\left(2+\sqrt{3}\right)\right)$, where p and q are rational numbers.

(6)

(a) $\frac{dy}{dx} = x$: $s = \int_0^{\sqrt{3}} \sqrt{1 + x^2}\, dx$ M1

$x = \sinh u$, $dx = \cosh u\, du$, $\sqrt{1 + \sinh^2 u} = \cosh u$: $\int \cosh^2 u\, du$ M1

Limits: $x = 0 \to u = 0$; $x = \sqrt{3} \to \cosh u = 2$, $u = \ln(\sqrt{3} + 2)$ A1

$\int \frac{1}{2}(1 + \cosh 2u)\, du = \frac{1}{2}[u + \sinh u \cosh u]$ M1

$= \frac{1}{2}(\ln(2 + \sqrt{3}) + \sqrt{3} \times 2) = \boxed{\sqrt{3} + (1/2)\ln(2 + \sqrt{3})}$ A1

(b) $S = 2\pi\int y\, ds = 2\pi\int_0^{\sqrt{3}} \frac{1}{2}x^2\sqrt{1 + x^2}\, dx = \pi\int_0^{\sqrt{3}} x^2\sqrt{1 + x^2}\, dx$ M1

$= \pi\int_0^{\ln(2 + \sqrt{3})} \sinh^2 u \cosh u \times \cosh u\, du = \pi\int \sinh^2 u \cosh^2 u\, du$ A1

$\sinh u \cosh u = \frac{1}{2}\sinh 2u$: $\sinh^2 u \cosh^2 u = \frac{1}{4}\sinh^2 2u = \frac{1}{8}(\cosh 4u - 1)$ M1

$S = \frac{\pi}{8}[\frac{1}{4}\sinh 4u - u]_0^{\ln(2 + \sqrt{3})}$ M1

$\sinh 2u = 2\sqrt{3} \times 2 = 4\sqrt{3}$, $\cosh 2u = 1 + 2(3) = 7$, $\sinh 4u = 2 \times 4\sqrt{3} \times 7 = 56\sqrt{3}$ M1

$S = \frac{\pi}{8}(14\sqrt{3} - \ln(2 + \sqrt{3})) = \boxed{\pi(7\sqrt{3}/4 - (1/8)\ln(2 + \sqrt{3}))}$ A1

(Total for Question 7 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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