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INTERNATIONAL A LEVEL MATHEMATICS • PURE 3 (WMA13)

Pure 3 Practice Paper 3

Worked Solutions

AIMING FOR A/A*

Written in the style of the Pearson Edexcel International Advanced Level paper (WMA13/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen — the working a full-marks answer shows, with a short note on why each step is taken

M1 A1 — the mark earned on that line: M = method, A = accuracy, B = independent mark

boxed — the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	Inverse tangent equation and proof	9	2
2	Stationary points in a geometric series	10	3
3	Quadratic in e^x, inverse, modulus	10	5
4	Compound angles and area between curves	9	7
5	Differentiating $x = e^y - e^{-y}$	10	9
6	Harmonic form and double angles	9	10
7	Roots by iteration and sign change	9	11
8	Algebraic division, area and proof	9	13

Question paper & mark scheme | Exam questions by topic | More practice papers

1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Solve the equation

$$\arctan x + \arctan 2x = \frac{\pi}{4}$$

(6)

(b) Prove that

$$2\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

(3)

(a) Let A = arctan x, B = arctan 2x, so A + B = $\frac{\pi}{4}$. Take tan of both sides:

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x + 2x}{1 - 2x^2} = \tan\frac{\pi}{4} = 1$ M1

$\frac{3x}{1-2x^2} = 1 \Rightarrow 3x = 1 - 2x^2$ A1

$2x^2 + 3x - 1 = 0 \Rightarrow x = (-3 \pm \sqrt{9+8}) \div 4$ M1

$x = (-3 + \sqrt{17}) \div 4 = 0.281\ldots$ or $x = (-3 - \sqrt{17}) \div 4 = -1.781\ldots$ A1

Check $x = -1.781\ldots$: both arctans are negative, $A + B = -1.059\ldots - 1.297\ldots = -\frac{3\pi}{4}$

$\tan(-\frac{3\pi}{4}) = 1$ too, but $A + B \neq \frac{\pi}{4}$: reject this root B1

$\boxed{x = (\sqrt{17} - 3)/4 \text{ only}}$ A1

(b) $\tan(2\arctan\frac{1}{3}) = \frac{2 \times 1/3}{1 - 1/9} = \frac{2}{3} \div \frac{8}{9} = \frac{3}{4}$ M1

$\tan(\text{sum}) = \frac{3/4 + 1/7}{1 - (3/4)(1/7)} = \frac{25}{28} \div \frac{25}{28} = 1$ M1

Range: $0 < \arctan\frac{1}{3} < \frac{\pi}{4}$, $0 < \arctan\frac{1}{7} < \frac{\pi}{4}$, so $0 < \text{sum} < \frac{3\pi}{4}$: sum = $\frac{\pi}{4}$ ✓ A1

(Total for Question 1 is 9 marks)

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2. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

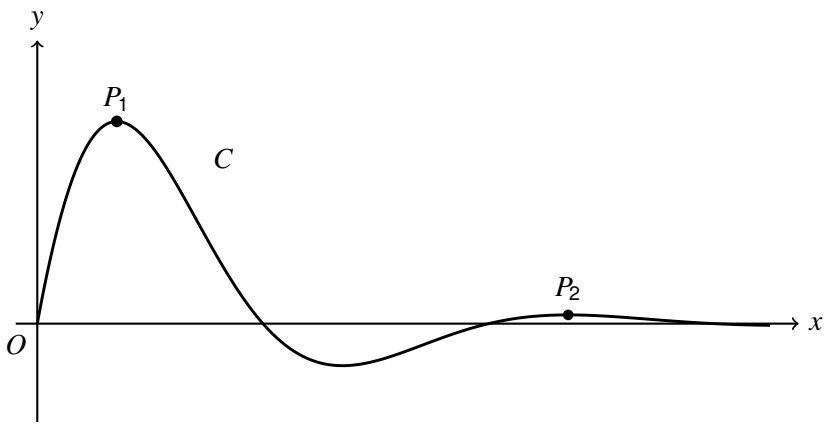

Figure 1

Figure 1 shows part of the curve C with equation

$$y = e^{-\frac{1}{2}x} \sin x \qquad x \geqslant 0$$

The maximum points on C, in order of increasing x, are $P_1, P_2, P_3, \ldots$

(a) Show that the x coordinate of every stationary point on C satisfies the equation $\tan x = 2$

(3)

(b) Prove that the y coordinates of $P_1, P_2, P_3, \ldots$ form a geometric sequence, and find its common ratio.

(4)

(c) Find the exact value of the sum of the y coordinates of all the maximum points on C.

(3)

(a) Product rule: $\frac{dy}{dx} = -\frac{1}{2}e^{-x/2} \sin x + e^{-x/2} \cos x$ M1

$= e^{-x/2}(\cos x - \frac{1}{2} \sin x)$ A1

$e^{-x/2} > 0$, so $\cos x = \frac{1}{2} \sin x \Rightarrow \frac{\sin x}{\cos x} = 2 \Rightarrow \tan x = 2$ ✓ A1

(b) $\tan x = 2$ when $x = \alpha + k\pi$, where $\alpha = \arctan 2 = 1.107\ldots$ M1

$\sin(\alpha + k\pi)$ is positive for k even and negative for k odd: maxima at $x = \alpha + 2n\pi$ M1

$y_n = e^{-(\alpha + 2n\pi)/2} \sin\alpha = e^{-\alpha/2} \sin\alpha \times (e^{-\pi})^n$, $n = 0, 1, 2, \ldots$ A1

Each height = the one before $\times e^{-\pi}$: geometric, common ratio $e^{-\pi}$ A1

(Not $e^{-\pi/2}$: $\tan x = 2$ every π, but only every other point is a maximum.)

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Question 2 continued

(c) $\tan\alpha = 2$, α acute: right triangle with sides 1, 2, $\sqrt{5}$, so $\sin\alpha = 2/\sqrt{5}$ M1

$|r| = e^{-\pi} < 1$: $S = \frac{a}{1-r}$ with $a = (2/\sqrt{5})e^{-(1/2)\arctan 2}$ M1

$S = 2e^{-(1/2)\arctan 2} \div [\sqrt{5}(1 - e^{-\pi})]$ $(= 0.537\ldots)$ A1

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(Total for Question 2 is 10 marks)

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3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(x) = e^{2x} - 6e^{x} + 5 \qquad x \in \mathbb{R}$$

(a) Sketch the curve with equation $y = f(x)$, showing the coordinates of the minimum point and of each point where the curve meets the coordinate axes, and the equation of the asymptote.

(3)

(b) Explain why f does not have an inverse.

The function g is defined by $g(x) = f(x)$, $x \geqslant k$, and g has an inverse.

Find the least possible value of k and, for this value of k, find $g^{-1}(x)$, stating its domain.

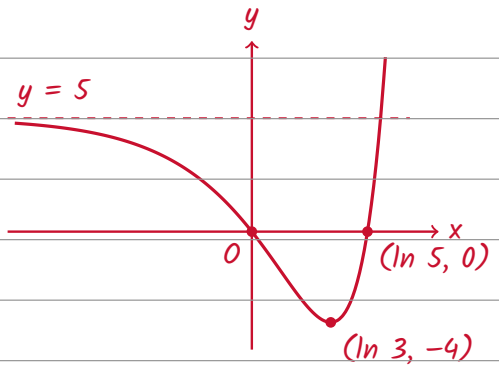
(4)

(c) Find the exact solutions of the equation $|f(x)| = 3$

(3)

(a) $f(x) = (e^x - 1)(e^x - 5)$: zero at $x = 0$ and $x = \ln 5$. $f'(x) = 2e^{2x} - 6e^x = 0 \Rightarrow e^x = 3$

Minimum $f(\ln 3) = 9 - 18 + 5 = -4$. As $x \to -\infty$, $e^x \to 0$ so $f(x) \to 5$.



B1 B1 B1

(b) $f(0) = f(\ln 5) = 0$: f is many-to-one, so f has no inverse B1

g must be one-to-one: start at the minimum. $k = \ln 3$ B1

$y = (e^x - 3)^2 - 4 \Rightarrow e^x - 3 = +\sqrt{y + 4}$ $(e^x \geqslant 3) \Rightarrow x = \ln(3 + \sqrt{y + 4})$ M1

$g^{-1}(x) = \ln(3 + \sqrt{x + 4})$, $x \geqslant -4$ A1

(c) $f(x) = -3$: $e^{2x} - 6e^x + 8 = 0 \Rightarrow (e^x - 2)(e^x - 4) = 0$ M1

$x = \ln 2$, $x = \ln 4$ A1

$f(x) = 3$: $e^{2x} - 6e^x + 2 = 0 \Rightarrow e^x = 3 \pm \sqrt{7}$, both positive $(3 - \sqrt{7} = 0.35\ldots)$

$x = \ln(3 + \sqrt{7})$, $x = \ln(3 - \sqrt{7})$ (four solutions in all) A1

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Question 3 continued

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(Total for Question 3 is 10 marks)

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4. In this question you must show all stages of your working.

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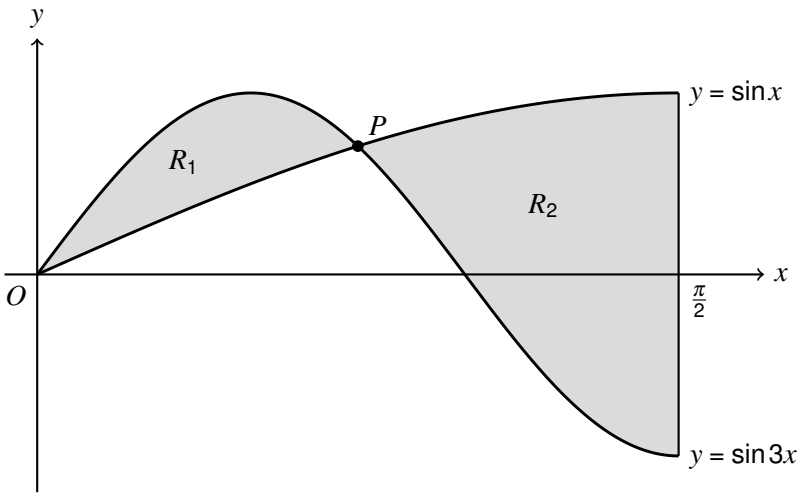

Figure 2

Figure 2 shows the curves with equations $y = \sin 3x$ and $y = \sin x$, for $0 \leqslant x \leqslant \dfrac{\pi}{2}$

The curves meet at O and at the point P.

The regions R_1 and R_2, shown shaded in Figure 2, are bounded by the two curves and the line $x = \frac{\pi}{2}$

(a) Use the compound angle formulae to find the exact x coordinate of P.

(3)

(b) Find the exact total area of R_1 and R_2, giving your answer in the form $a\sqrt{2}+b$, where a and b are rational.

(6)

(a) $\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = 2 \sin x \cos^2 x + (1 - 2\sin^2 x) \sin x$

$= 2 \sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3 \sin x - 4\sin^3 x$ M1

$3 \sin x - 4\sin^3 x = \sin x \Rightarrow \sin x(2 - 4\sin^2 x) = 0$. At P, $\sin x \neq 0$, so $\sin^2 x = \frac{1}{2}$ M1

$\sin x = 1/\sqrt{2}$ $(0 < x < \frac{\pi}{2})$ $x = \pi/4$ A1

(b) R_1: $\sin 3x$ is on top, from 0 to $\frac{\pi}{4}$: $\int (\sin 3x - \sin x)\, dx$ M1

$= [-\frac{1}{3}\cos 3x + \cos x]$ $(\cos\frac{3\pi}{4} = -\sqrt{2}/2,\ \cos\frac{\pi}{4} = \sqrt{2}/2)$ M1

$R_1 = (\sqrt{2}/6 + \sqrt{2}/2) - (-\frac{1}{3} + 1) = 2\sqrt{2}/3 - \frac{2}{3}$ A1

R_2: the curves swap at P, so $\sin x$ is on top, from $\frac{\pi}{4}$ to $\frac{\pi}{2}$: $\int (\sin x - \sin 3x)\, dx$ M1

$R_2 = [-\cos x + \frac{1}{3}\cos 3x] = (0 + 0) - (-\sqrt{2}/2 - \sqrt{2}/6) = 2\sqrt{2}/3$ A1

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Question 4 continued

Total = $R_1 + R_2$ $\boxed{(4/3)\sqrt{2} - 2/3}$ A1

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(Total for Question 4 is 9 marks)

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5. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The curve C has equation

$$x = e^{y} - e^{-y} \qquad x \in \mathbb{R}$$

(a) Find $\dfrac{dy}{dx}$ as a function of x, giving your answer in its simplest form.

(5)

(b) Find y as a function of x.

(3)

(c) Find an equation of the normal to C at the point where $y = \ln 2$, giving your answer in the form $y = mx + c$, where m and c are exact constants.

(2)

(a) $\frac{dx}{dy} = e^{y} + e^{-y}$. To write this in x, compare the squares: M1

$(e^{y} + e^{-y})^2 = e^{2y} + 2 + e^{-2y} = (e^{2y} - 2 + e^{-2y}) + 4 = (e^{y} - e^{-y})^2 + 4$ M1

So $(\frac{dx}{dy})^2 = x^2 + 4$ A1

$e^{y} + e^{-y} > 0$, so take the positive root: $\frac{dx}{dy} = \sqrt{x^2 + 4}$ M1

$dy/dx = 1/\sqrt{x^2 + 4}$ A1

(b) Multiply $x = e^{y} - e^{-y}$ by e^{y}: $e^{2y} - xe^{y} - 1 = 0$, a quadratic in e^{y} M1

$e^{y} = \frac{1}{2}(x \pm \sqrt{x^2 + 4})$ M1

$\sqrt{x^2 + 4} > |x|$, so the minus sign makes $e^{y} < 0$: reject. $y = \ln(\frac{1}{2}(x + \sqrt{x^2 + 4}))$ A1

(c) $y = \ln 2$: $x = 2 - \frac{1}{2} = \frac{3}{2}$, gradient $1 \div \sqrt{9/4 + 4} = \frac{2}{5}$, normal: $y - \ln 2 = -\frac{5}{2}(x - \frac{3}{2})$ M1

$y = -(5/2)x + 15/4 + \ln 2$ A1

(Total for Question 5 is 10 marks)

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6. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Given that

$$\cos x + \sin x = \frac{6}{5} \qquad 0 < x < \frac{\pi}{2}$$

(a) By writing $\cos x + \sin x$ in the form $R\sin(x+\alpha)$, sketch the curve with equation $y = \cos x + \sin x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$, and explain why there are exactly two values of x for which $\cos x + \sin x = \frac{6}{5}$

(4)

(b) Find the exact value of $\sin 2x$

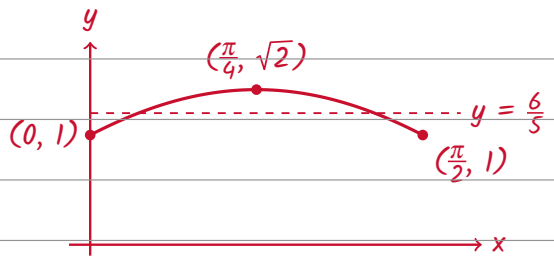
(2)

(c) Find the exact values of $\cos 2x$, stating which value goes with which value of x.

(3)

(a) $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$: $R\cos\alpha = 1$, $R\sin\alpha = 1$ M1

$R = \sqrt{2}$, $\alpha = \frac{\pi}{4}$: $\cos x + \sin x = \sqrt{2}\sin(x+\frac{\pi}{4})$ A1



B1

$1 < \frac{6}{5} < \sqrt{2}$: the line $y = \frac{6}{5}$ cuts the arch twice, once each side of $x = \frac{\pi}{4}$ B1

(b) Square both sides: $\cos^2x + 2\sin x\cos x + \sin^2x = \frac{36}{25} \Rightarrow 1 + \sin 2x = \frac{36}{25}$ M1

$\sin 2x = 11/25$ A1

(c) $\cos^2 2x = 1 - \frac{121}{625} = \frac{504}{625} \Rightarrow \cos 2x = \pm\sqrt{504} \div 25 = \pm 6\sqrt{14} \div 25$ M1 A1

Solution with $x < \frac{\pi}{4}$: $2x < \frac{\pi}{2}$, so $\cos 2x > 0$ $\cos 2x = 6\sqrt{14}/25$

Solution with $x > \frac{\pi}{4}$: $\frac{\pi}{2} < 2x < \pi$, so $\cos 2x < 0$ $\cos 2x = -6\sqrt{14}/25$ A1

(Total for Question 6 is 9 marks)

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7. In this question you must show all stages of your working.

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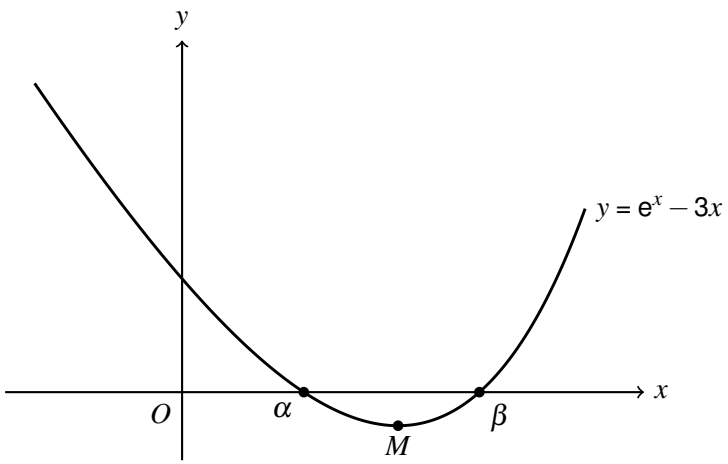

Figure 3

Figure 3 shows part of the curve with equation $y = e^x - 3x$

The curve has a minimum point M and meets the x-axis at the points where $x = \alpha$ and $x = \beta$

(a) Find the exact coordinates of M, and hence explain why the equation $e^x - 3x = 0$ has exactly two roots.

(3)

A student uses the iteration formula

$$x_{n+1} = \frac{1}{3}e^{x_n}$$

with $x_1 = 1.5$ to try to find β.

(b) Find x_2, x_3 and x_4 to 4 decimal places, and explain why this will not give β.

(2)

(c) Find β correct to 3 decimal places, justifying your answer.

(4)

(a) $\frac{dy}{dx} = e^x - 3 = 0 \Rightarrow x = \ln 3,\ y = 3 - 3\ln 3$ M1

M(ln 3, 3 − 3 ln 3) A1

ln 3 > 1, so $y_M < 0$, while $y(0) = 1 > 0$ and $y(2) = e^2 - 6 > 0$.

The curve only falls then rises, so exactly one root on each side of M ✓ A1

(b) $x_2 = 1.4939,\ x_3 = 1.4848,\ x_4 = 1.4714$ B1

Falling away from β: heading down to α, so it never gives β B1

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Question 7 continued

(c) Use $e^x = 3x \Rightarrow x = \ln(3x)$, $x_{n+1} = \ln(3x_n)$, $x_1 = 1.5$: M1

$x_2 = 1.5041$, $x_3 = 1.5068$, $x_4 = 1.5086$, ... (slowly rising to 1.512...) M1

Bounds for 3 d.p.: $f(1.5115) = -0.00097... < 0$, $f(1.5125) = 0.00056... > 0$ M1

Sign change, f continuous, so $1.5115 < \beta < 1.5125$ $\boxed{\beta = 1.512 \text{ (3 d.p.)}}$ A1

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(Total for Question 7 is 9 marks)

8. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(x) = \frac{x^3 + 2x^2 + 3x + 2}{x^2 + 1} \qquad x \in \mathbb{R}$$

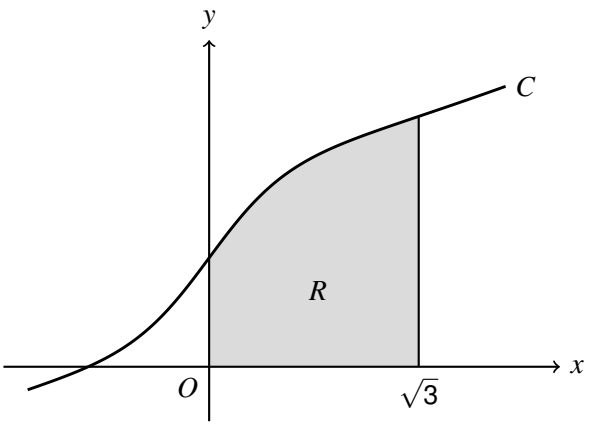

Figure 4

Figure 4 shows part of the curve C with equation $y = f(x)$

The region R, shown shaded in Figure 4, is bounded by C, the coordinate axes and the line $x = \sqrt{3}$

(a) Find the exact area of R, giving your answer in the form $p + q\sqrt{3} + \ln r$, where p, q and r are rational.

(6)

(b) Prove that f is an increasing function.

(3)

(a) Divide first: $x^3 + 2x^2 + 3x + 2 = (x + 2)(x^2 + 1) + 2x$ M1

$f(x) = x + 2 + \frac{2x}{x^2 + 1}$ A1

$\frac{2x}{x^2 + 1}$ is f'/f form: $\int \frac{2x}{x^2 + 1}\,dx = \ln(x^2 + 1)$ M1

$\int f(x)\,dx = \frac{1}{2}x^2 + 2x + \ln(x^2 + 1)$ M1

From 0 to $\sqrt{3}$: $(\frac{3}{2} + 2\sqrt{3} + \ln 4) - (0 + 0 + \ln 1)$ M1

Area of $R = 3/2 + 2\sqrt{3} + \ln 4$ A1

(b) Quotient rule on $\frac{2x}{x^2 + 1}$: $f'(x) = 1 + [2(x^2 + 1) - 2x \times 2x] \div (x^2 + 1)^2$ M1

$= [(x^2 + 1)^2 + 2 - 2x^2] \div (x^2 + 1)^2 = (x^4 + 2x^2 + 1 + 2 - 2x^2) \div (x^2 + 1)^2$

$f'(x) = (x^4 + 3) \div (x^2 + 1)^2$ A1

$x^4 + 3 > 0$ and $(x^2 + 1)^2 > 0$ for all x, so $f'(x) > 0$: f is increasing A1

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Question 8 continued

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TOTAL FOR PAPER IS 75 MARKS

END

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