



eClassroom

A-LEVEL MATHEMATICS • PURE 4

Differentiation

Worksheet

Pearson Edexcel International A Level (WMA14) & A Level (9MA0) • Calculator allowed: show every step

Free Differentiation worksheet for A-Level and International A-Level Maths, with full worked solutions



Tap a section to jump straight to it

By the end of this worksheet you can: differentiate parametric and implicit equations • find tangents and normals in the form $ax + by + c = 0$ • link rates of change with the chain rule • *Stretch:* differentiate a^x , find stationary points, and form differential equations.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Normals, stationary points and gradient conditions on parametric and implicit curves, a^x , and rates you set up yourself.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	11
Stretch marks	16	14	17	8
My score				

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A

Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE

Secure the method

Worked example A1: parametric differentiation

$x = t^2 + 1$, $y = t^3 - 2t$. Find the equation of the tangent at the point where $t = 2$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3t^2 - 2}{2t}; \text{ at } t = 2 \text{ this is } \frac{10}{4} = \frac{5}{2}$$

The point is (5, 4): $y - 4 = \frac{5}{2}(x - 5)$

$$2y = 5x - 17$$

Your turn A1

$x = 2t + 3$, $y = t^3 - 3t$. Find the equation of the tangent at the point where $t = 2$

Worked example A2: implicit differentiation

Find the gradient of $x^2 + 3xy - y^2 = 3$ at the point (1, 1)

Differentiate every term: $3xy$ needs the product rule, y^2 gives $2y \frac{dy}{dx}$

$$2x + 3y + 3x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

$$\text{At } (1, 1): 5 + (3 - 2) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -5$$

Your turn A2

Find the gradient of $x^2 - 2xy + 4y^2 = 12$ at the point (-2, 1)

Worked example A3: connected rates of change

The area of a circular ripple increases at $6 \text{ cm}^2 \text{ s}^{-1}$. Find $\frac{dr}{dt}$ when $r = 3 \text{ cm}$

Link the rates: $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$, with $A = \pi r^2$

$$6 = 2\pi(3) \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{\pi} \text{ cm s}^{-1}$$

Your turn A3

The volume of a sphere increases at $10 \text{ cm}^3 \text{ s}^{-1}$. Find $\frac{dr}{dt}$ when $r = 5 \text{ cm}$



STRETCH

Worked example A4: a parametric normal

$x = 3 \sin t$, $y = \cos 2t$. Find the normal at $t = \frac{\pi}{6}$ in the form $ax + by + c = 0$

$$\frac{dy}{dx} = -\frac{4}{3} \sin t = -\frac{2}{3} \text{ at } t = \frac{\pi}{6} : \text{normal gradient } \frac{3}{2}$$

Point $\left(\frac{3}{2}, \frac{1}{2}\right)$: $y - \frac{1}{2} = \frac{3}{2} \left(x - \frac{3}{2}\right)$

$$6x - 4y - 7 = 0$$

Your turn A4

$x = 4 \cos t$, $y = \sin 2t$. Find the gradient of the curve, and of the normal, at $t = \frac{\pi}{3}$

Worked example A5: differentiating a^x

Find $\frac{dy}{dx}$ at $(0, 1)$ on the curve $y 3^x = x + y^2$

$$\frac{d}{dx}(3^x) = 3^x \ln 3, \text{ with the product rule:}$$

$$3^x \frac{dy}{dx} + y 3^x \ln 3 = 1 + 2y \frac{dy}{dx}$$

At $(0, 1)$: $\frac{dy}{dx} + \ln 3 = 1 + 2 \frac{dy}{dx}$

$$\frac{dy}{dx} = \ln 3 - 1$$

Your turn A5

Find $\frac{dy}{dx}$ at $(1, 1)$ on the curve $2^y + x^2 = 3x$

Worked example A6: forming an equation

A tank has constant cross-section 2 m^2 . Water leaks at a rate proportional to $\sqrt{h}$, the depth. Form an equation for $\frac{dh}{dt}$

$$\frac{dV}{dt} = -k\sqrt{h} \text{ and } V = 2h: 2 \frac{dh}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = -\frac{k}{2}\sqrt{h}$$

Your turn A6

An oil slick's radius r grows at a rate inversely proportional to r ; $\frac{dr}{dt} = 0.5$ when $r = 10$. Show that $\frac{dA}{dt}$ is constant.

Worked example A7: stationary points

Find the points on $x^2 + xy + y^2 = 12$ where the tangent is horizontal.

$$\frac{dy}{dx} = 0 \text{ when } 2x + y = 0, \text{ so } y = -2x$$

$$x^2 - 2x^2 + 4x^2 = 12, \text{ so } x = \pm 2$$

$$(2, -4) \text{ and } (-2, 4)$$

Your turn A7

Find the points on $x^2 - xy + y^2 = 3$ where the tangent is horizontal.



B**Fluency**

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE*Secure the method*

B1 Find $\frac{dy}{dx}$ in terms of t for each curve.

(a) $x = t^2$, $y = 4t$

(b) $x = 3t + 1$, $y = t^3$

(c) $x = \cos t$, $y = \sin t$

(d) $x = e^t$, $y = te^t$

[4 marks]

B2 Find $\frac{dy}{dx}$ in terms of x and y for each curve.

(a) $x^2 + y^2 = 25$

(b) $y^3 + x = 2y$

(c) $xy + y^2 = 4x$

(d) $e^y + x^2y = 3$

[4 marks]

B3 Find the gradient of each curve at the given point.

(a) $x = t^2 - 1$, $y = 2t^3$, where $t = 1$

(b) $x = 2 \sin t$, $y = \cos 2t$, where $t = \frac{\pi}{6}$

(c) $x^2 + y^2 = 10$, at $(1, 3)$

(d) $x^3 + y^3 = 9$, at $(2, 1)$

[4 marks]

B4 (a) Find the tangent to $x^2 + 2y^2 = 9$ at $(1, 2)$, in the form $ax + by = c$

(b) Find the normal to $x^2 + 2y^2 = 9$ at $(1, 2)$

(c) Find the tangent to $x = t^2$, $y = t^3$ at the point where $t = 2$

(d) Find the normal to $x = t^2$, $y = t^3$ at the point where $t = 2$, in the form $ax + by = c$

[4 marks]

B5 (a) $A = \pi r^2$ and $\frac{dr}{dt} = 0.2$. Find $\frac{dA}{dt}$ when $r = 5$

(b) A cube has volume $V = x^3$ and $\frac{dV}{dt} = 12$. Find $\frac{dx}{dt}$ when $x = 2$

(c) A sphere has surface area $S = 4\pi r^2$ and $\frac{dr}{dt} = 0.1$. Find $\frac{dS}{dt}$ when $r = 3$

[3 marks]



B6 Differentiate with respect to x

(a) 5^x

(b) 2^{3x}

(c) $x3^x$

(d) 4^x , and find its gradient at $x = \frac{1}{2}$ in terms of $\ln 2$

[4 marks]

STRETCH

B7 (a) For $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, find $\frac{dy}{dx}$ as a single fraction in t

(b) Find the points on the curve in (a) where the tangent is vertical.

(c) For $x = \ln t$, $y = t^2 - 2t$, $t > 0$, find the coordinates of the stationary point.

(d) For $x = \sin t$, $y = \sin 3t$, find the gradient where $t = \frac{\pi}{4}$

[4 marks]

B8 (a) Find the gradient of $x^2y + y^3 = 10$ at $(3, 1)$

(b) Find the gradient of $\ln y + xy = 2$ at $(2, 1)$

(c) Find the points on $x^2 + y^2 - 6x = 0$ where the tangent is vertical.

(d) By taking logarithms, differentiate $y = x^x$, $x > 0$, and find the gradient at $x = 1$

[4 marks]

B9 (a) A cone has $h = 2r$, so $V = \frac{2}{3}\pi r^3$, and $\frac{dV}{dt} = 6$. Find $\frac{dr}{dt}$ when $r = 3$

(b) A ladder has $x^2 + y^2 = 25$. Given $\frac{dx}{dt} = 0.5$, find $\frac{dy}{dt}$ when $x = 3$ and $y = 4$

(c) $y = \ln(1 + x^2)$ and $\frac{dx}{dt} = 2$. Find $\frac{dy}{dt}$ when $x = 1$

(d) $A = 10 \sin \theta$ and $\frac{d\theta}{dt} = 0.1$. Find $\frac{dA}{dt}$ when $\theta = \frac{\pi}{3}$

[4 marks]

B10 Write a differential equation for each situation, using k for a positive constant.

(a) The temperature T falls at a rate proportional to $T - 20$

(b) A number N of bacteria increases at a rate proportional to $\sqrt{N}$

(c) A sphere of radius r loses volume at a rate proportional to its surface area. Find $\frac{dr}{dt}$

(d) A tank of cross-section 3 m^2 gains $0.6 \text{ m}^3/\text{min}$ and loses $0.2h \text{ m}^3/\text{min}$. Find $\frac{dh}{dt}$

[4 marks]



C

Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE

Secure the method

- C1** For $x^2 + y^3 = 7$, Adam writes $2x + 3y^2 = 0$
 Explain his error, and find the correct expression for $\frac{dy}{dx}$ [2 marks]
- C2** For $x^2 + 3xy = 10$, Bree differentiates $3xy$ as $3x \frac{dy}{dx}$
 Explain her error, and find the correct expression for $\frac{dy}{dx}$ [2 marks]
- C3** For $x = t^2$, $y = 4t - t^3$, Caleb writes $\frac{dy}{dx} = \frac{2t}{4 - 3t^2}$ and finds the gradient at $t = 1$ is 2
 Find his error, and give the correct gradient at $t = 1$ [3 marks]
- C4** A sphere's radius increases at 0.1 cm s^{-1} . Della writes " $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = 4\pi r^2 = 36\pi$ when $r = 3$ "
 Explain her error, and find the correct rate. [3 marks]

STRETCH

- C5** The tangent to a curve at $(2, 3)$ has gradient $\frac{2}{3}$. Ewan writes the normal as " $2x - 3y + 5$ "
 Find his two errors, and give the normal in the form $ax + by + c = 0$ [3 marks]
- C6** For a curve with $\frac{dy}{dx} = \frac{x - 2y}{2x - y}$, Faye finds the stationary points by solving $2x - y = 0$
 Explain her error, and say what the points with $2x - y = 0$ are. [3 marks]
- C7** Gil writes $\frac{d}{dx}(8^x) = x8^{x-1}$
 Explain his error, give the correct derivative, and find the exact gradient of $y = 8^x$ at $x = \frac{1}{3}$ in terms of $\ln 2$ [3 marks]
- C8** The curve C has parametric equations $x = t^2$, $y = t^3$
 (a) Show that the tangent at the point with parameter p , $p \neq 0$, has equation $2y = 3px - p^3$ (2)
 (b) Show that this tangent meets C again at the point with parameter $-\frac{p}{2}$ (3)
 [5 marks]



D

Problem solving

You decide the method. Read the whole question first, and show every step.

CORE

Secure the method

D1 A curve C has equation

$$x^3 + xy + y^2 = 13$$

The point $P(1, 3)$ lies on C . Using implicit differentiation, find an expression for $\frac{dy}{dx}$ in terms of x and y , and

hence show that the gradient of C at the point P is $-\frac{6}{7}$. You must show each step of your working.

[3 marks]

D2 In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 1 shows the curve C with parametric equations

$$x = 4 \cos t, \quad y = 3 \sin t$$

The tangent l at the point P , where $t = \frac{\pi}{4}$, meets the x -axis at A and the y -axis at B

(a) Show that l has equation $3x + 4y = 12\sqrt{2}$

(3)

(b) Find the exact area of the triangle OAB , where O is the origin.

(1)

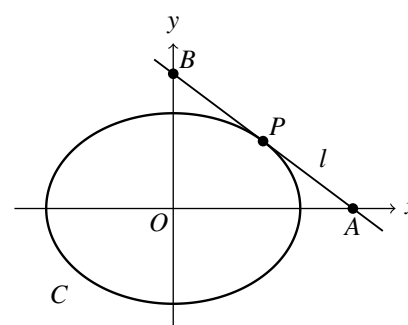


Figure 1

[4 marks]

D3 Figure 2 shows a hemispherical bowl of radius 6 cm. When the depth of water is h cm, the volume of water is

$$V = \frac{\pi h^2(18 - h)}{3} \text{ cm}^3$$

Water is poured in at a constant rate of $10\pi \text{ cm}^3$ per second. Find the rate at which the depth of the water is increasing at the instant when $h = 2$, giving your answer in cm per second.

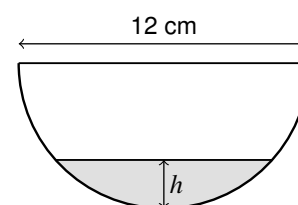


Figure 2

[3 marks]

D4 A curve C has equation

$$4^x + y^2 = 5y$$

The point $P(1, 1)$ lies on C . Using implicit differentiation, find an expression for $\frac{dy}{dx}$ in terms of x and y , and

hence show that the gradient of C at the point P is $\frac{8}{3} \ln 2$

[3 marks]



STRETCH

D5 Figure 3 shows the curve C with parametric equations

$$x = \ln(t+2), \quad y = t^2 - 2t, \quad t > -2$$

The curve has a minimum point M

(a) Find $\frac{dy}{dx}$ in terms of t , as a product of factors.

(b) Find the exact coordinates of M , and use part (a) to justify that M is a minimum point of C

(2)

(2)

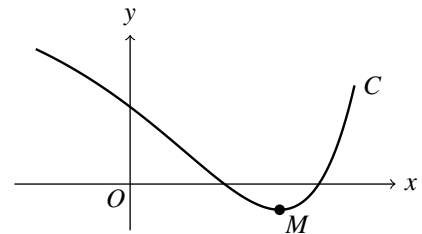


Figure 3

[4 marks]

D6 Figure 4 shows the curve C with equation

$$x^2 - 2xy + 3y^2 = 8$$

At the points A and B , the tangent to C is parallel to the line $y = x$

(a) Find $\frac{dy}{dx}$ in terms of x and y

(b) Using your answer to part (a), find the exact coordinates of the points A and B

(2)

(2)

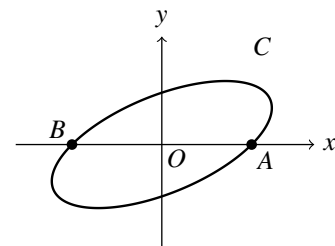


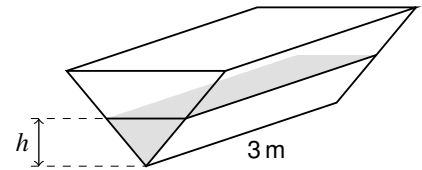
Figure 4

[4 marks]



STRETCH *continued*

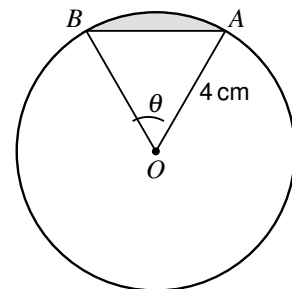
- D7** Figure 5 shows a trough of length 3 m. Its cross-section is an isosceles triangle, and when the depth of water is h m the width of the water surface is $2h$ m. Water flows in at a constant rate of 0.06 m^3 per minute.

**Figure 5**

- (a) Show that the volume of water is $V = 3h^2 \text{ m}^3$ (2)
- (b) Find the rate at which the depth of water is increasing when $h = 0.2$, stating your units. (2)

[4 marks]

- D8** Figure 6 shows a circle with centre O and radius 4 cm. The chord AB subtends an angle θ radians at O . The area of the shaded segment is $A \text{ cm}^2$. The angle θ increases at a constant rate of 0.05 radians per second.

**Figure 6**

- (a) Show that $\frac{dA}{d\theta} = 8(1 - \cos \theta)$ (1)
- (b) Find $\frac{dA}{dt}$ when $\theta = \frac{\pi}{3}$ (2)
- (c) Find θ , $0 < \theta < \pi$, when A is increasing at 0.6 cm^2 per second.

(2)

[5 marks]



E**Exam practice**

Exam-style questions in the style of WMA14. Write your answers in the spaces, and show every step.

CORE

Exam questions for everyone

- E1** *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

A curve C has equation

$$y^3 + 2xy = x^2 + 11$$

The point $P(1, 2)$ lies on C

- (a) Find $\frac{dy}{dx}$ in terms of x and y , simplifying your answer.

(3)

- (b) Find an equation of the normal to C at P , giving your answer in the form $ax + by + c = 0$, where a , b and c are integers to be found.

(2)

(Total for Question E1 is 5 marks)



- E2** Sand is poured onto a horizontal floor at a constant rate of $9\pi \text{ cm}^3$ per second, forming a cone whose height h cm is always equal to the radius r cm of its base, as shown in Figure 7.

(a) Find the rate at which the height of the cone is increasing when $h = 3$

(3)

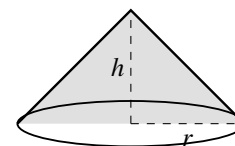


Figure 7

(b) The curved surface area of a cone is $S = \pi r l$, where l is its slant height. Find the exact rate at which S is increasing when $h = 3$, giving your answer in terms of π and stating its units.

(3)

(Total for Question E2 is 6 marks)



STRETCH

- E3** In this question you must show all stages of your working.
Solutions relying on calculator technology are not acceptable.

Figure 8 shows the curve C with parametric equations

$$x = t^2, \quad y = 2t$$

The tangents at P ($t = 2$) and Q ($t = -\frac{1}{2}$) meet at R

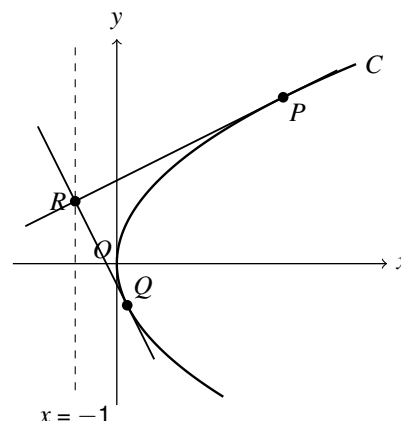


Figure 8

- (a) Show that the tangent to C at the point with parameter t has equation $ty = x + t^2$
- (3)
- (b) Find the coordinates of R , and show that the tangents at P and Q are perpendicular.
- (3)
- (c) Prove that whenever the tangents at the points with parameters p and q are perpendicular, they meet on the line $x = -1$

(2)

(Total for Question E3 is 8 marks)

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