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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 1 (WFM01)

Further Pure 1 Practice Paper 3

Worked Solutions

AIMING FOR A/A*

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM01/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	A quartic with unknown coefficients	10	2
2	Roots of a quadratic with a parameter	10	3
3	A series from n to $2n$	11	4
4	Numerical methods near an asymptote	11	5
5	A transformation with a given area	11	6
6	A normal to a parabola and a locus	11	7
7	Induction for a two-step recurrence	11	9

Question paper & mark scheme | Exam questions by topic | More practice papers

1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(z) = z^4 + az^2 + bz + 10$$

where a and b are real constants. Given that $1 + 2i$ is a root of the equation $f(z) = 0$

(a) find the value of a and the value of b

(5)

(b) find the other three roots of the equation $f(z) = 0$

(3)

The four roots are represented by the points A, B, C and D on an Argand diagram.

(c) Find the area of the quadrilateral formed by A, B, C and D

(2)

(a) $(1 + 2i)^2 = 1 + 4i + 4i^2 = -3 + 4i$ $\quad (1 + 2i)^4 = (-3 + 4i)^2 = -7 - 24i$ M1 A1

Substitute into $f(z) = 0$: $-7 - 24i + a(-3 + 4i) + b(1 + 2i) + 10 = 0$

Real parts: $3 - 3a + b = 0$ $\quad$ Imaginary parts: $-24 + 4a + 2b = 0$ M1

$b = 3a - 3$, so $4a + 2(3a - 3) = 24 \Rightarrow 10a = 30$ M1

$\boxed{a = 3,\ b = 6}$ A1

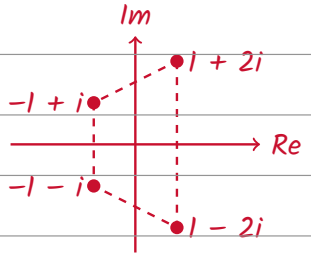
(b) Real coefficients, so the conjugate $\boxed{1 - 2i}$ is a root B1

$(z - 1 - 2i)(z - 1 + 2i) = (z - 1)^2 + 4 = z^2 - 2z + 5$ is a factor

$z^4 + 3z^2 + 6z + 10 = (z^2 - 2z + 5)(z^2 + cz + 2)$: z^3 term: $c - 2 = 0$, so $c = 2$ M1

Check z: $5c - 4 = 6$ ✓. $z^2 + 2z + 2 = 0 \Rightarrow (z + 1)^2 = -1 \Rightarrow \boxed{z = -1 \pm i}$ A1

(c) Plot the four roots and join them in order round the shape:



A trapezium: parallel sides 4 and 2, a distance 2 apart. Area $= \frac{1}{2}(4 + 2) \times 2 = \boxed{6}$ M1 A1

(Total for Question 1 is 10 marks)

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2. The equation

$$x^2 + kx + k + 6 = 0$$

where k is a constant, has **real** roots α and β

(a) Find an expression for $\alpha^2 + \beta^2$ in terms of k

(2)

Given that $\alpha^2 + \beta^2 = 12$

(b) find the value of k, giving a reason for rejecting any other value.

(5)

(c) Using this value of k, find a quadratic equation, with integer coefficients, whose roots are α^3 and β^3

(3)

(a) $\alpha + \beta = -k$, $\alpha\beta = k + 6$. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = k^2 - 2(k + 6)$ M1

$\boxed{k^2 - 2k - 12}$ A1

(b) $k^2 - 2k - 12 = 12 \Rightarrow k^2 - 2k - 24 = 0$ M1

$(k - 6)(k + 4) = 0 \Rightarrow k = 6$ or $k = -4$ A1

The roots are real, so $b^2 - 4ac \geqslant 0$: $k^2 - 4(k + 6) \geqslant 0$. Test both values: M1

$k = 6$: $36 - 48 = -12 < 0$ (not real) $k = -4$: $16 + 16 - 24 = 8 \geqslant 0$ ✓ A1

$\boxed{k = -4}$ ($k = 6$ is rejected because it gives non-real roots) A1

(c) $k = -4$: $x^2 - 4x + 2 = 0$, so $\alpha + \beta = 4$, $\alpha\beta = 2$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = 64 - 3 \times 2 \times 4 = 40$ M1 A1

$\alpha^3\beta^3 = (\alpha\beta)^3 = 8$: $x^2 - 40x + 8 = 0$ $\boxed{x^2 - 40x + 8 = 0}$ A1

(Total for Question 2 is 10 marks)

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3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Use the standard results for $\sum_{r=1}^{m} r$ and $\sum_{r=1}^{m} r^2$ to find $\sum_{r=1}^{m} r(r+1)$, giving your answer in fully factorised form.

(3)

(b) Hence show that, for all positive integers n,

$$\sum_{r=n}^{2n} r(r+1) = \tfrac{1}{3}n(n+1)(7n+5)$$

(4)

(c) Find the value of n for which

$$\sum_{r=n}^{2n} r(r+1) = \sum_{r=1}^{n} (28r-15)$$

giving a reason for rejecting any other solutions.

(4)

(a) $\sum r(r+1) = \sum r^2 + \sum r = \frac{1}{6}m(m+1)(2m+1) + \frac{1}{2}m(m+1)$ M1

$= \frac{1}{6}m(m+1)[(2m+1)+3] = \frac{1}{6}m(m+1)(2m+4)$ M1

$\boxed{\frac{1}{3}m(m+1)(m+2)}$ A1

(b) The sum starts at $r = n$, so subtract the sum up to $n-1$ (not up to n): M1

$m = 2n$: $\frac{1}{3}(2n)(2n+1)(2n+2)$ $\quad m = n-1$: $\frac{1}{3}(n-1)n(n+1)$ A1

First term $= \frac{4}{3}n(n+1)(2n+1)$, so difference $= \frac{1}{3}n(n+1)[4(2n+1)-(n-1)]$ M1

$= \frac{1}{3}n(n+1)(8n+4-n+1) = \frac{1}{3}n(n+1)(7n+5)$ as required A1

(c) $\sum (28r-15) = 28 \times \frac{1}{2}n(n+1) - 15n = 14n^2 + 14n - 15n = 14n^2 - n$ B1

$\frac{1}{3}n(n+1)(7n+5) = n(14n-1)$. $n \neq 0$, so divide by n and $\times 3$: $7n^2 + 12n + 5 = 42n - 3$ M1

$7n^2 - 30n + 8 = 0 \Rightarrow (n-4)(7n-2) = 0$ A1

$n = 4$ or $n = \frac{2}{7}$; n must be a positive integer, so reject $\frac{2}{7}$ (and 0): $\boxed{n = 4}$ A1

Check $n = 4$: $20 + 30 + 42 + 56 + 72 = 220$ and $28 \times 10 - 60 = 220$ ✓

(Total for Question 3 is 11 marks)

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4.

$$f(x) = x^2 - 4 + \frac{1}{x-1}, \qquad x \neq 1$$

(a) Show that f(0.5) and f(1.5) have opposite signs, and explain why this does **not** show that the equation $f(x) = 0$ has a root in the interval [0.5, 1.5]

(3)

(b) Show that the equation $f(x) = 0$ has a root α in the interval $[-2.1, -2]$

(2)

(c) Taking -2 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(3)

A student instead applies the Newton–Raphson process once to $f(x)$ with a first approximation of 1.6

(d) Find the second approximation that the student obtains, giving your answer to 3 decimal places, and explain why 1.6 is not a suitable first approximation.

(3)

(a) $f(0.5) = 0.25 - 4 + \frac{1}{-0.5} = -5.75$ $f(1.5) = 2.25 - 4 + \frac{1}{0.5} = 0.25$ opposite signs M1 A1

But f is not continuous on [0.5, 1.5]: it has an asymptote at x = 1 (jumps from $-\infty$ to $+\infty$) B1

(b) $f(-2.1) = 4.41 - 4 + \frac{1}{-3.1} = 0.0874...$ $f(-2) = 4 - 4 + \frac{1}{-3} = -\frac{1}{3}$ M1

Sign change, and f is continuous on $[-2.1, -2]$ ($x = 1$ is not in it), so α is in $[-2.1, -2]$ A1

(c) $f'(x) = 2x - (x-1)^{-2}$, $f'(-2) = -4 - \frac{1}{9} = -\frac{37}{9}$ B1

$x_1 = -2 - \frac{f(-2)}{f'(-2)} = -2 - (-\frac{1}{3}) \div (-\frac{37}{9}) = -2 - \frac{3}{37} = -2.0810...$ M1

$\boxed{-2.081}$ A1

(d) $f(1.6) = 2.56 - 4 + \frac{1}{0.6} = 0.2266...$ $f'(1.6) = 3.2 - \frac{1}{0.36} = 0.4222...$

$x_1 = 1.6 - \frac{0.2266...}{0.4222...} = 1.0631...$ M1

$\boxed{1.063}$ A1

f'(1.6) is close to 0: the tangent is nearly flat, so x_1 jumps to near the asymptote x = 1 B1

(Also f(x) > 0 for all x > 1, so there is no root near 1.6 to find.)

(Total for Question 4 is 11 marks)

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5.

$$\mathbf{P} = \begin{pmatrix} k & k+2 \\ -1 & 3 \end{pmatrix}$$

where k is a constant.

(a) Find $\det \mathbf{P}$ in terms of k, and hence find the value of k for which $\mathbf{P}$ is singular.

(2)

The triangle T has vertices $O(0, 0)$, $A(3, 0)$ and $B(0, 2)$. The image of T under the transformation represented by $\mathbf{P}$ has area 30

(b) Find the possible values of k

(4)

(c) For the negative value of k, find the coordinates of the images of A and B

(2)

The transformation represented by $\mathbf{P}$, with this negative value of k, is followed by a rotation through 90° anticlockwise about (0, 0). The combined transformation is represented by the matrix $\mathbf{R}$

(d) Find the coordinates of the point that is mapped by $\mathbf{R}$ onto the point (5, 5)

(3)

(a) $\det P = 3k - (-1)(k + 2) = 3k + k + 2 = \boxed{4k + 2}$ B1

Singular when $4k + 2 = 0$: $\boxed{k = -\frac{1}{2}}$ B1

(b) Area of $T = \frac{1}{2} \times 3 \times 2 = 3$, so the area scale factor is $|\det P| = \frac{30}{3} = 10$ M1

The determinant can be negative: $4k + 2 = 10$ or $4k + 2 = -10$ M1

$\boxed{k = 2}$ or $\boxed{k = -3}$ A1 A1

(c) $k = -3$: $P = \begin{pmatrix} -3 & -1 \\ -1 & 3 \end{pmatrix}$. $P\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -9 \\ -3 \end{pmatrix}$, $P\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

$\boxed{A'(-9, -3)}$ $\boxed{B'(-2, 6)}$ B1 B1

(d) Rotation 90° anticlockwise: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. P first, so R = (rotation) × P: M1

$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -3 & -1 \end{pmatrix}$ A1

$\det R = (1)(-1) - (-3)(-3) = -1 - 9 = -10$. Undo R with R^{-1}:

$R^{-1}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} 10 \\ 20 \end{pmatrix}$ $\boxed{(-1, -2)}$ A1

Check: $R\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 + 6 \\ 3 + 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ ✓

(Total for Question 5 is 11 marks)

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6.

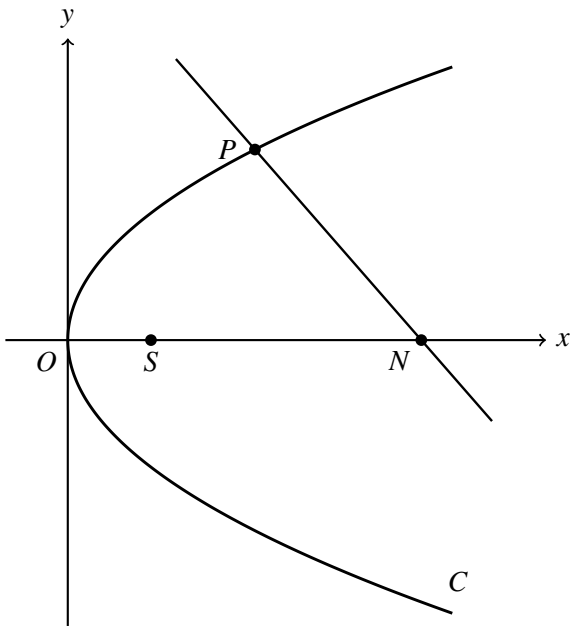

Figure 1

The parabola C has equation $y^2 = 8x$ and focus S. The point $P\left(2p^2, 4p\right)$, $p \neq 0$, lies on C. The normal to C at P meets the x-axis at the point N, as shown in Figure 1.

(a) Show that an equation of the normal to C at P is

$$y + px = 2p^3 + 4p$$

(4)

(b) Find, in terms of p, the area of the triangle SPN when $p > 0$

(3)

(c) Given that the area of the triangle SPN is 40, find the possible coordinates of P

(2)

The point M is the midpoint of PN

(d) Show that, as p varies, M lies on a parabola, and find a Cartesian equation of this parabola.

(2)

(a) $y^2 = 8x$: $2y\frac{dy}{dx} = 8$, $\frac{dy}{dx} = \frac{4}{y} = \frac{4}{4p} = \frac{1}{p}$ at P M1 A1

Normal gradient $= -p$: $y - 4p = -p(x - 2p^2)$ M1

$y - 4p = -px + 2p^3 \Rightarrow y + px = 2p^3 + 4p$ as required A1

(b) $y = 0$: $px = 2p^3 + 4p \Rightarrow x = 2p^2 + 4$, so $N(2p^2 + 4, 0)$; focus $S(2, 0)$ $(a = 2)$ B1

Base $SN = 2p^2 + 4 - 2 = 2p^2 + 2$, height $= 4p$ $(p > 0)$: Area $= \frac{1}{2}(2p^2 + 2)(4p)$ M1

$4p(p^2 + 1)$ A1

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Question 6 continued

(c) $p < 0$ gives height $-4p$, so Area $= 4|p|(p^2 + 1) = 40 \Rightarrow |p|^3 + |p| - 10 = 0$ M1

$|p| = 2$ $(8 + 2 = 10)$, so $p = 2$ or $p = -2$: $(8, 8)$ and $(8, -8)$ A1

(d) $M = \left(\frac{2p^2 + 2p^2 + 4}{2}, \frac{4p + 0}{2}\right) = (2p^2 + 2, 2p)$. $y = 2p \Rightarrow p = \frac{y}{2}$ M1

$x = 2\left(\frac{y}{2}\right)^2 + 2 = \frac{y^2}{2} + 2 \Rightarrow y^2 = 2x - 4$, which is a parabola A1

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(Total for Question 6 is 11 marks)

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7. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 5, \qquad u_2 = 13, \qquad u_{n+2} = 5u_{n+1} - 6u_n$$

(a) Prove by induction that, for all positive integers n,

$$u_n = 2^n + 3^n$$

(6)

(b) Using the result in part (a), prove by induction that u_{2m-1} is divisible by 5 for all positive integers m

(5)

(a) $n = 1$: $2 + 3 = 5 = u_1$ ✓ $n = 2$: $4 + 9 = 13 = u_2$ ✓ (two base cases) B1

Assume true for $n = k$ AND $n = k + 1$: $u_k = 2^k + 3^k$, $u_{k+1} = 2^{k+1} + 3^{k+1}$ M1

$u_{k+2} = 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k) = 10 \times 2^k + 15 \times 3^k - 6 \times 2^k - 6 \times 3^k$ M1

$= 4 \times 2^k + 9 \times 3^k$ A1

$= 2^{k+2} + 3^{k+2}$, the result for $n = k + 2$ A1

True for $n = 1, 2$; true for $n = k, k + 1 \Rightarrow$ true for $n = k + 2$; so true for all $n \geqslant 1$. A1

(b) Let $f(m) = u_{2m-1} = 2^{2m-1} + 3^{2m-1}$. $m = 1$: $f(1) = 2 + 3 = 5$, divisible by 5 B1

Assume 5 divides $f(k)$. $f(k + 1) = 2^{2k+1} + 3^{2k+1} = 4 \times 2^{2k-1} + 9 \times 3^{2k-1}$ M1

$= 4(2^{2k-1} + 3^{2k-1}) + 5 \times 3^{2k-1} = 4f(k) + 5 \times 3^{2k-1}$ (take out $4f(k)$, not $f(k)$) A1

$4f(k)$ and $5 \times 3^{2k-1}$ are both divisible by 5, so $f(k + 1)$ is divisible by 5 A1

True for $m = 1$, and if true for $m = k$ then true for $m = k + 1$, so true for all $m \geqslant 1$. A1

(Total for Question 7 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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