Integration

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 4

Question 1 — Worked Solution

u=2x+1 ⇒ x=(u–1)/2, dx=du/2

$$\int x(2x+1)^5\,\mathrm{d}x = \int \frac{u-1}{2} \cdot u^5 \cdot \frac{du}{2} = \frac{1}{4}\int (u^6 - u^5)\,\mathrm{d}u$$

$$= \frac{1}{4}\left(\frac{u^7}{7} - \frac{u^6}{6}\right) + c$$

$$\therefore = \frac{(2x+1)^7}{28} - \frac{(2x+1)^6}{24} + c$$

Question 2 — Worked Solution

$u=x^2 \Rightarrow du=2x\,dx$

$$\int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}\int e^u\,\mathrm{d}u = \frac{1}{2}e^u + c$$

$$\therefore = \frac{e^{x^2}}{2} + c$$

Question 3 — Worked Solution

$u=\sin x \Rightarrow du=\cos x\,dx$; $\cos^2 x=1-u^2$

$$\int \sin x\cos^3 x\,dx = \int u(1-u^2)\,du = \frac{u^2}{2}-\frac{u^4}{4}+c$$

$$\therefore = \frac{\sin^2 x}{2}-\frac{\sin^4 x}{4}+c \quad \left(=-\frac{\cos^4 x}{4}+c\right)$$

Question 4 — Worked Solution

$u=\cos x \Rightarrow du=-\sin x\,dx$; $\sin^2 x=1-u^2$

$$\int \sin^3 x\,\mathrm{d}x = \int \sin x(1-\cos^2 x)\,\mathrm{d}x = \int -(1-u^2)\,\mathrm{d}u$$

$$= -u + \frac{u^3}{3} + c$$

$$\therefore = -\cos x + \frac{\cos^3 x}{3} + c$$

Question 5 — Worked Solution

$u=1+\cos x \Rightarrow du=-\sin x\, dx$

Limits: $x=0 \Rightarrow u=2$; $x=\pi/2 \Rightarrow u=1$

$$\int_0^{\pi/2} \frac{\sin x}{(1+\cos x)^2}\,\mathrm{d}x = \int_2^1 \frac{-1}{u^2}\,\mathrm{d}u = \int_1^2 u^{-2}\,\mathrm{d}u$$

$$= \left[-\frac{1}{u}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2} \quad \checkmark$$

Question 6 — Worked Solution

Write lnx = lnx·1, integrate by parts: u=lnx, dv=1

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + c$$

$$\therefore = x \ln x - x + c$$

Question 7 — Worked Solution

u=3x, dv=sin2x dx ⇒ du=3dx, v=–cos2x/2

$$\int 3x\sin 2x\,dx = -\frac{3x\cos 2x}{2} + \int \frac{3\cos 2x}{2}\,dx$$

$$\therefore = -\frac{3x\cos 2x}{2} + \frac{3\sin 2x}{4} + c$$

Question 8 — Worked Solution

Integration by parts twice

Let I = $\int e^x \cos x\, dx$. Set $u = \cos x$, $dv = e^x dx$:

$$I = e^x \cos x + \int e^x \sin x\, dx$$

Apply parts again to $\int e^x \sin x\, dx$ with $u = \sin x$, $dv = e^x dx$:

$$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx = e^x \sin x - I$$

Substitute back:

$$I = e^x \cos x + e^x \sin x - I \;\Rightarrow\; 2I = e^x(\cos x + \sin x)$$

$$\therefore I = \frac{e^x(\sin x + \cos x)}{2} + c \quad \checkmark$$

Question 9 — Worked Solution

First application: $u=x^2$, $dv=e^{2x}dx$

$$\int x^2e^{2x}\,\mathrm{d}x = \frac{x^2e^{2x}}{2} - \int xe^{2x}\,\mathrm{d}x$$

Second application: $u=x$, $dv=e^{2x}dx$

$$\int xe^{2x}\,\mathrm{d}x = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$$

Combine:

$$\therefore = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + c = \frac{e^{2x}(2x^2-2x+1)}{4} + c$$

Question 10 — Worked Solution

u=lnx, dv=x^{3}dx ⇒ du=1/x dx, v=x^4/4

$$\int x^3 \ln x\,dx = \frac{x^4 \ln x}{4} - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx = \frac{x^4 \ln x}{4} - \frac{x^4}{16} + c$$

$$\therefore = \frac{x^4}{16}(4\ln x - 1) + c$$

Question 11 — Worked Solution

First: $u=x^2$, $dv=\cos x\,dx$

$$\int x^2 \cos x \, dx = x^2 \sin x - 2\int x \sin x \, dx$$

Second: $u=x$, $dv=\sin x\,dx$

$$\int x \sin x \, dx = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x$$

Combine:

$$\therefore = x^2 \sin x + 2x\cos x - 2\sin x + c$$

Question 12 — Worked Solution

(a) Partial fractions

$$\frac{5}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

x=–1: 5=–3A ⇒ A=–5/3. x=2: 5=3B ⇒ B=5/3

$$\therefore = \frac{-5/3}{x+1} + \frac{5/3}{x-2}$$

(b) Show integral result

$$\int \frac{5}{(x+1)(x-2)}\,\mathrm{d}x = -\frac{5}{3}\ln|x+1| + \frac{5}{3}\ln|x-2| + c$$

$$= \frac{5}{3}\ln\left|\frac{x-2}{x+1}\right| + c \quad \checkmark$$

Question 13 — Worked Solution

(a) Partial fractions

$$\frac{x+5}{x(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x+5 \equiv A(x+1)^2 + Bx(x+1) + Cx$$

x=0: 5=A. x=–1: 4=–C ⇒ C=–4. Compare x^2: 0=A+B ⇒ B=–5

$$\therefore = \frac{5}{x} - \frac{5}{x+1} - \frac{4}{(x+1)^2}$$

(b) Show integral result

$$\int \frac{x+5}{x(x+1)^2}\,dx = 5\ln|x| - 5\ln|x+1| + \frac{4}{x+1} + c$$

$$= 5\ln\left|\frac{x}{x+1}\right| + \frac{4}{x+1} + c \quad ✓$$

Question 14 — Worked Solution

$x=t^2$, $y=2t$; $dx/dt=2t$

$$\text{Area} = \int_0^2 y\frac{dx}{dt}\,\mathrm{d}t = \int_0^2 2t\cdot 2t\,\mathrm{d}t = \int_0^2 4t^2\,\mathrm{d}t$$

$$= \left[\frac{4t^3}{3}\right]_0^2 = \frac{32}{3}$$

$$\therefore \text{Area} = \frac{32}{3}$$

Question 15 — Worked Solution

$x=t^2$, $y=t-t^3$; $dx/dt=2t$

$y=t(1-t^2) \geq 0$ for $0 \leq t \leq 1$ ✓

$$\text{Area} = \int_0^1 y\frac{dx}{dt}\,\mathrm{d}t = \int_0^1 (t-t^3)\cdot 2t\,\mathrm{d}t = \int_0^1 (2t^2-2t^4)\,\mathrm{d}t$$

$$= \left[\frac{2t^3}{3} - \frac{2t^5}{5}\right]_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{10-6}{15}$$

$$\therefore = \frac{4}{15}$$

Question 16 — Worked Solution

$$V = \pi\int_0^3 y^2\,dx = \pi\int_0^3 (x+1)\,dx$$

$$= \pi\left[\frac{x^2}{2} + x\right]_0^3 = \pi\left(\frac{9}{2} + 3\right)$$

$$\therefore = \frac{15\pi}{2}$$

Question 17 — Worked Solution

$$V = \pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi} = \frac{\pi}{2}(\pi - 0)$$

$$\therefore = \frac{\pi^2}{2}$$

Question 18 — Worked Solution

$$V = \pi \int_0^1 (e^x + 1)^2 \, dx = \pi \int_0^1 (e^{2x} + 2e^x + 1) \, dx$$

$$= \pi \left[\frac{e^{2x}}{2} + 2e^x + x \right]_0^1$$

$$= \pi \left[\left(\frac{e^2}{2} + 2e + 1 \right) - \left(\frac{1}{2} + 2 + 0 \right) \right]$$

$$= \pi \left(\frac{e^2}{2} + 2e + 1 - \frac{5}{2} \right)$$

$$\therefore = \pi \left(\frac{e^2}{2} + 2e - \frac{3}{2} \right) \quad \checkmark$$

Question 19 — Worked Solution

Separate variables

$$\int \frac{1}{y}\,\mathrm{d}y = \int x\,\mathrm{d}x \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + c$$

y=2 when x=0: ln2=c

$$\ln y = \frac{x^2}{2} + \ln 2$$

$$\therefore y = 2e^{x^2/2}$$

Question 20 — Worked Solution

Separate variables

$$\int \frac{1}{1+y^2}\,\mathrm{d}y = \int \frac{1}{1+x^2}\,\mathrm{d}x$$

$$\arctan y = \arctan x + c$$

y=1 when x=0: π/4=0+c ⇒ c=π/4

$$\arctan y = \arctan x + \frac{\pi}{4}$$

$$\therefore y = \tan\left(\arctan x + \frac{\pi}{4}\right)$$

Question 21 — Worked Solution

(a) Show T = 20 + 60·(1/2)$^{t/10}$

$$\int \frac{dT}{T-20} = -k\int dt \Rightarrow \ln|T-20| = -kt + c$$

$$T - 20 = Ae^{-kt}$$

T=80 at t=0: A=60.   T=50 at t=10: 30=60e^{–10k} ⇒ e^{–10k}=1/2 ⇒ k=ln2/10

$$T = 20 + 60e^{-t\ln 2/10} = 20 + 60 \cdot 2^{-t/10} = 20 + 60 \cdot \left(\frac{1}{2}\right)^{t/10} \quad ✓$$

(b) T at t=30

$$T(30) = 20 + 60 \cdot \left(\frac{1}{2}\right)^3 = 20 + \frac{60}{8} = 20 + 7.5$$

$$\therefore T = 27.5\ ^\circ\mathrm{C}$$

Question 22 — Worked Solution

Separate variables and use partial fractions

$$\int \frac{100\,dP}{P(100-P)} = \int dt$$

PF: 100/(P(100–P)) = 1/P + 1/(100–P)

$$\ln|P| - \ln|100-P| = t + c \;\Rightarrow\; \ln\left|\frac{P}{100-P}\right| = t + c$$

P=10 at t=0: ln(10/90)=c ⇒ c=–ln9

$$\frac{P}{100-P} = \frac{1}{9}e^{t}$$

$$9Pe^{-t} = 100 - P \;\Rightarrow\; P(1+9e^{-t}) = 100$$

$$\therefore P = \frac{100}{1+9e^{-t}}$$

Question 23 — Worked Solution

(a) Show k = ln(1.6)/5

P=500 at t=0: P=500e^{kt} ✓

P=800 at t=5: 800=500e^{5k} ⇒ e^{5k}=8/5=1.6

$$5k = \ln(1.6) \;\Rightarrow\; k = \frac{\ln(1.6)}{5} \quad ✓$$

(b) Find t when P=2000

$$2000 = 500e^{kt} \;\Rightarrow\; e^{kt} = 4 \;\Rightarrow\; t = \frac{\ln 4}{k} = \frac{5\ln 4}{\ln(1.6)}$$

$$\therefore t = \frac{5\ln 4}{\ln(1.6)} \approx 14.7 \text{ years}$$

Question 24 — Worked Solution

$M = M_0e^{-\lambda t}$. Half-life 8 years: $M_0/2 = M_0e^{-8\lambda} \Rightarrow \lambda = \ln 2/8$

$$M = 200e^{-t\ln 2/8} = 200 \cdot 2^{-t/8}$$

At t=20:

$$M = 200 \cdot 2^{-20/8} = 200 \cdot 2^{-5/2} = \frac{200}{4\sqrt{2}} = \frac{50}{\sqrt{2}} = \frac{50\sqrt{2}}{2}$$

$$\therefore M = 25\sqrt{2}\ \text{g}$$

Question 25 — Worked Solution

(a) Show T = 25 + 65·(7/13)$^{t/15}$

$$T - 25 = Ae^{-kt}$$

T=90 at t=0: A=65. T=60 at t=15: 35=65e^{-15k} ⇒ e^{-15k}=35/65=7/13 ⇒ k=ln(13/7)/15

$$T = 25 + 65e^{-kt} = 25 + 65 \cdot \left(\frac{7}{13}\right)^{t/15} \quad ✓$$

(b) T at t=45

$$T(45) = 25 + 65 \cdot \left(\frac{7}{13}\right)^3 = 25 + 65 \cdot \frac{343}{2197} = 25 + \frac{22295}{2197}$$

$$\therefore \approx 35.1\ ^\circ\text{C}$$