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INTERNATIONAL A LEVEL MATHEMATICS • PURE 2 (WMA12)

Pure 2 Practice Paper 3

Worked Solutions

AIMING FOR A/A*

Written in the style of the Pearson Edexcel International Advanced Level paper (WMA12/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	A binomial coefficient condition	8	2
2	Geometric series with a trigonometric ratio	9	3
3	Areas forming a geometric series	10	4
4	Two circles through two points	10	6
5	The largest triangle in a circle	10	7
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7	Logarithms of $\sin x$ and $\cos x$	10	10
8	Working back from the trapezium rule	9	11

Question paper & mark scheme | Exam questions by topic | More practice papers

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1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

In the expansion of

$$(x+3)(1+2x)^n$$

where n is a positive integer, the coefficient of x^3 is 540

(a) Find the value of n, showing that there is only one possible value.

(6)

(b) Find the coefficient of x^4 in the expansion.

(2)

(a) $(1+2x)^n = 1 + {}^nC_1(2x) + {}^nC_2(2x)^2 + {}^nC_3(2x)^3 + \ldots$

An x^3 term in $(x+3)(1+2x)^n$ comes from $3 \times (x^3$ term) and $x \times (x^2$ term): M1

$3 \times {}^nC_3 \times 2^3 + {}^nC_2 \times 2^2 = 540$ A1

Use ${}^nC_3 = \frac{1}{6}n(n-1)(n-2)$ and ${}^nC_2 = \frac{1}{2}n(n-1)$:

$4n(n-1)(n-2) + 2n(n-1) = 540$ M1

Take out $2n(n-1)$: $2n(n-1)(2n-3) = 540 \Rightarrow n(n-1)(2n-3) = 270$

Expand: $2n^3 - 5n^2 + 3n - 270 = 0$ A1

Try $n = 6$: $432 - 180 + 18 - 270 = 0$ ✓ so $(n-6)(2n^2 + 7n + 45) = 0$ M1

$2n^2 + 7n + 45$: $b^2 - 4ac = 49 - 360 < 0$, no real roots. So $\boxed{n = 6}$ is the only value. A1

(b) x^4 term: $3 \times {}^6C_4 \times 2^4 + {}^6C_3 \times 2^3 = 3 \times 15 \times 16 + 20 \times 8$ M1

$= 720 + 160 = \boxed{880}$ A1

(Total for Question 1 is 8 marks)

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2. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

A geometric series has first term 5 and common ratio $2\cos\theta$, where $0 \leqslant \theta < 360°$

The series has a sum to infinity, and the sum to infinity is $\frac{8}{3}$ times the sum of the second and third terms.

(a) Find the possible values of the common ratio.

(6)

(b) Hence find all the possible values of θ, giving your answers to one decimal place.

(3)

(a) Let $r = 2\cos\theta$. Sum to infinity: $S_\infty = \frac{5}{1-r}$. Second + third terms $= 5r + 5r^2$

$\frac{5}{1-r} = \frac{8}{3}(5r + 5r^2) \Rightarrow 15 = 40r(1 + r)(1 - r)$ M1

$3 = 8r(1 - r^2) = 8r - 8r^3 \Rightarrow 8r^3 - 8r + 3 = 0$ A1

Try $r = \frac{1}{2}$: $1 - 4 + 3 = 0$ ✓ so $(2r - 1)(4r^2 + 2r - 3) = 0$ M1

$4r^2 + 2r - 3 = 0$: $r = (-2 \pm \sqrt{4 + 48}) \div 8 = \frac{1}{4}(-1 \pm \sqrt{13})$ M1

Three roots: $r = \frac{1}{2}$, $r = 0.651\ldots$, $r = -1.151\ldots$ A1

A sum to infinity needs $|r| < 1$, so reject $-1.151\ldots$: $\boxed{r = \frac{1}{2} \text{ or } r = \frac{1}{4}(-1 + \sqrt{13})}$ B1

(Careful: $r = -1.151\ldots$ would give $\cos\theta = -0.576$, a real angle. It is rejected because $|r| > 1$.)

(b) $\cos\theta = \frac{r}{2}$: $\cos\theta = \frac{1}{4}$ or $\cos\theta = \frac{1}{8}(-1 + \sqrt{13}) = 0.3257\ldots$ M1

$\cos\theta = \frac{1}{4}$: $\theta = 75.52°$, $360° - 75.52° = 284.48°$ A1

$\cos\theta = 0.3257\ldots$: $\theta = 70.99°$, $289.01°$ $\boxed{\theta = 71.0°,\ 75.5°,\ 284.5°,\ 289.0°}$ A1

(Total for Question 2 is 9 marks)

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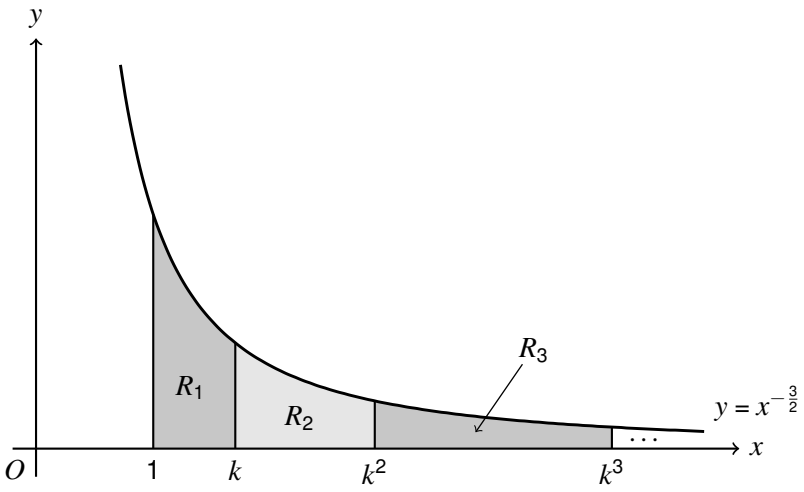

Figure 1

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 1 shows part of the curve with equation $y = x^{-\frac{3}{2}}$, $x > 0$

The constant k is greater than 1

For each positive integer n, the region R_n is bounded by the curve, the x-axis and the lines $x = k^{n-1}$ and $x = k^n$

The regions R_1, R_2 and R_3 are shown shaded in Figure 1.

(a) Show that the areas of $R_1, R_2, R_3, \ldots$ form a geometric sequence, and find its common ratio in terms of k.

(6)

(b) Show that the sum to infinity of the areas does not depend on the value of k.

(2)

(c) Given that $k = 4$, find the least value of n for which the total area of $R_1, R_2, \ldots, R_n$ exceeds 1.99

(2)

(a) $\int x^{-3/2}\,dx = \frac{x^{-1/2}}{-1/2} = -2x^{-1/2}$ M1

Area of R_n = $[-2x^{-1/2}]$ from k^{n-1} to k^n = $-2k^{-n/2} + 2k^{-(n-1)/2}$ M1

$= 2k^{-n/2}(k^{1/2} - 1)$ (since $k^{-(n-1)/2} = k^{-n/2} \times k^{1/2}$) A1

Area $R_{n+1} \div$ Area $R_n = [2k^{-(n+1)/2}(k^{1/2} - 1)] \div [2k^{-n/2}(k^{1/2} - 1)]$ M1

The factor $2(k^{1/2} - 1)$ cancels: $k^{-(n+1)/2} \div k^{-n/2} = k^{-1/2}$, the same for every n A1

So the areas form a geometric sequence with common ratio $k^{-1/2} = 1 \div \sqrt{k}$ A1

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Question 3 continued

(b) First term ($n = 1$): $2k^{-1/2}(k^{1/2} - 1) = 2(1 - k^{-1/2})$. $k > 1$, so $0 < k^{-1/2} < 1$: S_∞ exists M1

Sum to infinity $= 2(1 - k^{-1/2}) \div (1 - k^{-1/2}) = 2$, which does not depend on k A1

(c) $k = 4$: first term $2(1 - \frac{1}{2}) = 1$, ratio $\frac{1}{2}$. $2(1 - 0.5^n) > 1.99 \Rightarrow 0.5^n < 0.005$

$n \log 0.5 < \log 0.005 \Rightarrow n > \frac{\log 0.005}{\log 0.5} = 7.64...$ (log 0.5 < 0, so the sign turns round) M1

$\boxed{n = 8}$ (check: total after 7 is 1.984, after 8 is 1.992) A1

(Total for Question 3 is 10 marks)

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4. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The points A and B have coordinates (1, 1) and (7, 1) respectively.

(a) Show that there are exactly two circles of radius 4 that pass through both A and B, and find an equation for each circle.

(4)

(b) Sketch both circles on the same axes, showing the coordinates of their centres and the positions of A and B.

(2)

(c) Find the area of the region that lies inside both circles, giving your answer to 3 significant figures.

(4)

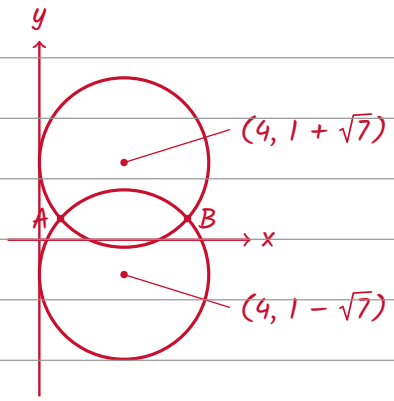
(a) The perpendicular from the centre bisects chord AB, so the centre is on $x = 4$: $(4, c)$ M1

The distance from $(4, c)$ to $A(1, 1)$ is the radius 4: $(4 - 1)^2 + (c - 1)^2 = 16 \Rightarrow (c - 1)^2 = 7$ M1

$c = 1 \pm \sqrt{7}$: two possible centres, so exactly two circles A1

$(x - 4)^2 + (y - 1 - \sqrt{7})^2 = 16$ and $(x - 4)^2 + (y - 1 + \sqrt{7})^2 = 16$ A1

(b) Both circles have radius 4 and centre on $x = 4$, so both touch the y-axis. B1 B1



(c) The region inside both circles is two segments, one on each side of AB.

AB = 6, so in each circle half of AB is 3: $\sin(\frac{1}{2}\theta) = \frac{3}{4} \Rightarrow \theta = 2\sin^{-1}(\frac{3}{4}) = 1.6961...$ radians M1

Segment $= \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2} \times 16 \times (1.6961... - 0.9922...) = 5.6317...$ M1

Two equal segments: $2 \times 5.6317... = 11.263...$ M1

Area = 11.3 (to 3 significant figures) A1

(Total for Question 4 is 10 marks)

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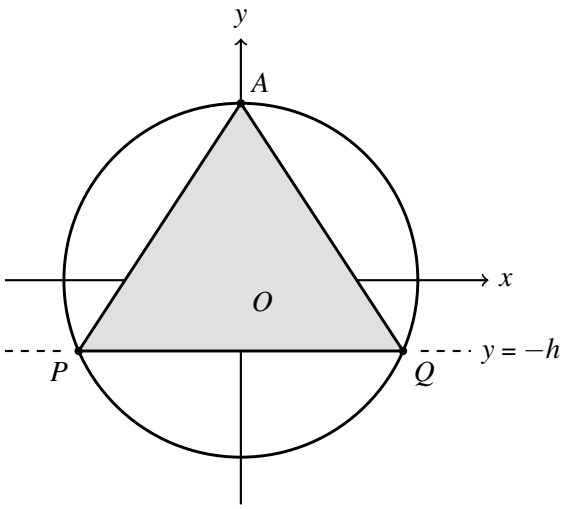

Diagram not drawn to scale

Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 2 shows the circle with equation $x^2 + y^2 = 9$

The point A has coordinates (0, 3).

The points P and Q lie on the circle and on the line with equation $y = -h$, where $0 < h < 3$

The triangle APQ is shown shaded in Figure 2.

(a) Find an expression, in terms of h, for the area of triangle APQ.

(3)

(b) Use calculus to find the exact maximum area of triangle APQ, justifying that your answer is a maximum.

(7)

(a) On $y = -h$: $x^2 + h^2 = 9 \Rightarrow x = \pm\sqrt{9 - h^2}$, so $PQ = 2\sqrt{9 - h^2}$ M1

The height of the triangle, from A(0, 3) down to $y = -h$, is $3 + h$ M1

Area $= \frac{1}{2} \times 2\sqrt{9 - h^2} \times (3 + h)$: $\boxed{A = (3 + h)\sqrt{9 - h^2}}$ A1

(b) A has a square root, and P2 has no chain rule. Since $A > 0$, A is greatest when A^2 is. M1

$A^2 = (3 + h)^2(9 - h^2) = (9 + 6h + h^2)(9 - h^2) = 81 + 54h - 6h^3 - h^4$ A1

$\frac{d(A^2)}{dh} = 54 - 18h^2 - 4h^3$ M1

$= 0 \Rightarrow 2h^3 + 9h^2 - 27 = 0$. $h = \frac{3}{2}$ works $(6.75 + 20.25 - 27 = 0)$: $(2h - 3)(h + 3)^2 = 0$ M1

$h = \frac{3}{2}$ ($h = -3$ is not in $0 < h < 3$) A1

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Question 5 continued

$\frac{d^2(A^2)}{dh^2} = -36h - 12h^2 = -54 - 27 = -81 < 0$ at $h = \frac{3}{2}$, so a maximum M1

$A = \frac{9}{2}\sqrt{9 - \frac{9}{4}} = \frac{9}{2} \times \frac{3}{2}\sqrt{3}$ maximum area $= \frac{27}{4}\sqrt{3}$ (an equilateral triangle) A1

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(Total for Question 5 is 10 marks)

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6. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Factorise $n^5 - n$ completely.

(2)

(b) Prove that, for every integer n, the number $n^5 - n$ is divisible by 30

(6)

(c) Show, by means of a counter-example, that $n^5 - n$ is not divisible by 60 for every integer n.

(1)

(a) $n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1)$ M1

$n^5 - n = n(n - 1)(n + 1)(n^2 + 1)$ A1

(b) $30 = 2 \times 3 \times 5$, so show it is divisible by 2, by 3 and by 5.

$n - 1$, n, $n + 1$ are three consecutive integers: one is even and one is a multiple of 3. B1

For 5, every integer n is $5k$, $5k + 1$, $5k - 1$, $5k + 2$ or $5k - 2$ for some integer k: M1

$n = 5k$: n is a multiple of 5. $n = 5k \pm 1$: $n \mp 1 = 5k$ is a multiple of 5. A1

$n = 5k \pm 2$: $n^2 + 1 = 25k^2 \pm 20k + 4 + 1$ M1

$= 5(5k^2 \pm 4k + 1)$, a multiple of 5 A1

In every case $n^5 - n$ has factors 2, 3 and 5 (no common factor between them),

so $n^5 - n$ is divisible by $2 \times 3 \times 5 = 30$ for every integer n. A1

(c) $n = 2$: $2^5 - 2 = 30$, which is not divisible by 60. B1

(Total for Question 6 is 9 marks)

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7. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Solve, for $0 < x < 2\pi$,

$$\log_2(\sin x) + \log_2(\cos x) = -2$$

giving your answers to 3 significant figures.

(7)

(b) Find the greatest value of the constant k for which the equation

$$\log_2(\sin x) + \log_2(\cos x) = k$$

has a solution.

(3)

(a) Add the logs: $\log_2(\sin x \cos x) = -2$ M1

$\sin x \cos x = 2^{-2} = \frac{1}{4}$ A1

Square, then write $\cos^2 x = 1 - \sin^2 x$: $\sin^2 x(1 - \sin^2 x) = \frac{1}{16}$ M1

$16\sin^4 x - 16\sin^2 x + 1 = 0$ A1

A quadratic in $\sin^2 x$: $\sin^2 x = (16 \pm \sqrt{192}) \div 32 = \frac{1}{4}(2 \pm \sqrt{3}) = 0.0670$ or 0.9330 M1

The logs need $\sin x > 0$ AND $\cos x > 0$, so x is in the first quadrant only. This also removes the extra roots that squaring made (they have $\sin x \cos x = -\frac{1}{4}$). B1

$\sin x = 0.2588\ldots \Rightarrow x = 0.2618\ldots$; $\sin x = 0.9659\ldots \Rightarrow x = 1.3090\ldots$ $\boxed{x = 0.262, 1.31}$ A1

(b) $\sin x \cos x = 2^k \Rightarrow \sin^2 x(1 - \sin^2 x) = 4^k$. Let $t = \sin^2 x$, with $0 < t < 1$: M1

$t(1 - t) = \frac{1}{4} - (t - \frac{1}{2})^2$, so its greatest value is $\frac{1}{4}$ (when $t = \frac{1}{2}$) M1

$4^k \leqslant \frac{1}{4} = 4^{-1} \Rightarrow k \leqslant -1$: the greatest value is $\boxed{k = -1}$ A1

(Total for Question 7 is 10 marks)

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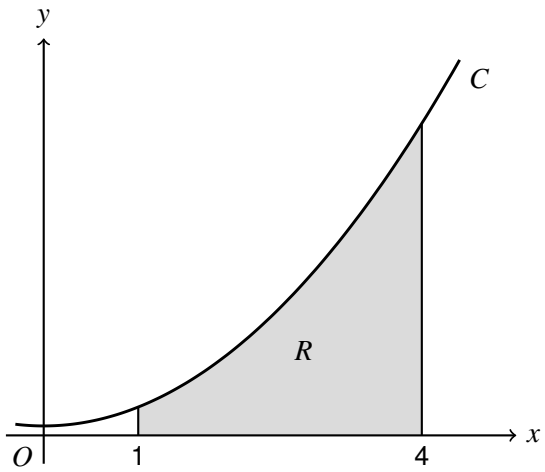

Figure 3

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 3 shows part of the curve C with equation

$$y = ax^2 + b$$

where a and b are positive constants.

The region R, shown shaded in Figure 3, is bounded by C, the x-axis and the lines $x = 1$ and $x = 4$

Using the trapezium rule with three strips of equal width gives an estimate of 46 for the area of R.

The exact area of R is 45

(a) Find the value of a and the value of b.

(5)

(b) Use the trapezium rule with six strips of equal width to estimate the area of R, and show that the error is one quarter of the error when three strips are used.

(3)

(c) Assuming that this pattern continues, write down the estimate given by the trapezium rule with twelve strips of equal width.

(1)

(a) Exact: $\int (ax^2 + b)\,dx$ from 1 to 4 $= \left[\frac{a}{3}x^3 + bx\right] = \left(\frac{64a}{3} + 4b\right) - \left(\frac{a}{3} + b\right)$ M1

$= 21a + 3b = 45$ A1

3 strips from 1 to 4: $h = 1$. $\frac{1}{2} \times 1 \times [(a + b) + (16a + b) + 2(4a + b) + 2(9a + b)]$ M1

$= \frac{1}{2}(43a + 6b) = 46 \Rightarrow 43a + 6b = 92$ A1

Double the first: $42a + 6b = 90$. Subtract: $a = 2$, then $3b = 3$: $\boxed{a = 2,\ b = 1}$ A1

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Question 8 continued

(b) 6 strips: $h = 0.5$. $y = 2x^2 + 1$ at $x = 1, 1.5, \ldots, 4$: 3, 5.5, 9, 13.5, 19, 25.5, 33 B1

$\frac{1}{2} \times 0.5 \times [3 + 33 + 2(5.5 + 9 + 13.5 + 19 + 25.5)] = 0.25 \times [36 + 145]$ M1

$= 45.25$. Error: $45.25 - 45 = 0.25$, and with 3 strips it was $46 - 45 = 1$. $0.25 = \frac{1}{4} \times 1$ ✓ A1*

(c) Doubling the strips again divides the error by 4: $0.25 \div 4 = 0.0625$, so 45.0625 B1

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(Total for Question 8 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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