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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 2 (WFM02)

Further Pure 2 Practice Paper 1

Worked Solutions

AIMING FOR B

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM02/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	An inequality with two fractions	6	2
2	Partial fractions and the method of differences	7	3
3	Fifth roots of a complex number	7	4
4	A first-order differential equation	8	5
5	A Maclaurin series and an estimate	8	6
6	Loci in the Argand diagram	8	7
7	A second-order differential equation	9	8
8	Tangents to a polar curve and an area	10	9
9	de Moivre's theorem and an integral	12	10

Question paper & mark scheme | Exam questions by topic | More practice papers

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1. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Use algebra to find the set of values of x for which

$$\frac{x}{x-2} < \frac{2x}{x+1}$$

(6)

Multiply both sides by $(x-2)^2(x+1)^2$, which is positive, so the inequality sign stays: M1

$x(x-2)(x+1)^2 < 2x(x+1)(x-2)^2$

Take out the common factor $x(x+1)(x-2)$: $x(x+1)(x-2)[(x+1)-2(x-2)] < 0$ M1

$x(x+1)(x-2)(5-x) < 0$ A1

Critical values: $x = -1,\ 0,\ 2,\ 5$ B1

Sketch the quartic $x(x+1)(x-2)(5-x)$: it is negative for $x < -1$, $0 < x < 2$, $x > 5$ dM1

$\boxed{x < -1 \quad \text{or} \quad 0 < x < 2 \quad \text{or} \quad x > 5}$ A1

(Check $x = 1$: $\frac{1}{-1} = -1 < \frac{2}{2} = 1$ ✓)

(Total for Question 1 is 6 marks)

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2. **(a)** Express $\dfrac{4}{(r+1)(r+3)}$ in partial fractions.

(2)

(b) Hence, using the method of differences, show that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{4}{(r+1)(r+3)} = \frac{n(5n+13)}{3(n+2)(n+3)}$$

(5)

(a) $\frac{4}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$; $4 = A(r+3) + B(r+1)$; $r = -1$: $A = 2$, $r = -3$: $B = -2$ M1

$2/(r+1) - 2/(r+3)$ A1

(b) $r = 1$: $\frac{2}{2} - \frac{2}{4}$ $r = 2$: $\frac{2}{3} - \frac{2}{5}$ $r = 3$: $\frac{2}{4} - \frac{2}{6}$ ... M1

$r = n-1$: $\frac{2}{n} - \frac{2}{n+2}$ $r = n$: $\frac{2}{n+1} - \frac{2}{n+3}$

Everything cancels except $\frac{2}{2} + \frac{2}{3} - \frac{2}{n+2} - \frac{2}{n+3}$ A1

$= \frac{5}{3} - \frac{2(n+3)+2(n+2)}{(n+2)(n+3)} = \frac{5}{3} - \frac{2(2n+5)}{(n+2)(n+3)}$ M1

Over the denominator $3(n+2)(n+3)$, the numerator is $5(n+2)(n+3) - 6(2n+5)$ M1

$= 5n^2 + 25n + 30 - 12n - 30 = 5n^2 + 13n = n(5n+13)$

So $\sum = \frac{n(5n+13)}{3(n+2)(n+3)}$ as required A1

(Total for Question 2 is 7 marks)

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3.

$$w = 16 + 16\sqrt{3}\,\mathrm{i}$$

(a) Express w in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

(2)

(b) Solve the equation $z^5 = w$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

(4)

(c) Show the points representing your answers to part (b) on a single Argand diagram.

(1)

(a) $|w| = \sqrt{16^2 + (16\sqrt{3})^2} = \sqrt{256 + 768} = 32$; $\arg w = \arctan\sqrt{3} = \frac{\pi}{3}$ (1st quadrant) M1

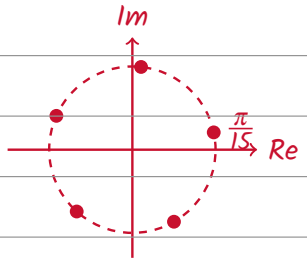
$w = 32e^{i\pi/3}$ A1

(b) $z^5 = 32e^{i(\pi/3 + 2k\pi)}$, so $z = 32^{1/5}\, e^{i(\pi/3 + 2k\pi)/5} = 2e^{i(\pi/15 + 2k\pi/5)}$ M1 B1

$k = 0, 1, 2, -1, -2$: $\theta = \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, -\frac{5\pi}{15}, -\frac{11\pi}{15}$ (all in $-\pi < \theta \leqslant \pi$) M1

$z = 2e^{-11\pi i/15}, 2e^{-\pi i/3}, 2e^{\pi i/15}, 2e^{7\pi i/15}, 2e^{13\pi i/15}$ A1

(c) Five points on a circle of radius 2, equally spaced at 72°:



B1

(Total for Question 3 is 7 marks)

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4. **(a)** Find the general solution of the differential equation

$$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x = \cos^3 x, \qquad 0 \leqslant x < \frac{\pi}{2}$$

giving your answer in the form $y = \mathrm{f}(x)$

(6)

Given that $y = 2$ when $x = 0$

(b) find the exact value of y when $x = \frac{\pi}{3}$

(2)

(a) Divide by cos x: $\frac{dy}{dx} + y \tan x = \cos^2 x$ M1

Integrating factor $= e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$ M1 A1

Multiply through: $\sec x \frac{dy}{dx} + y \sec x \tan x = \cos x$, so $\frac{d}{dx}(y \sec x) = \cos x$

$y \sec x = \int \cos x \, dx = \sin x + c$ M1 A1

$\boxed{y = \sin x \cos x + c \cos x}$ A1

(b) $x = 0, y = 2$: $2 = 0 + c$, so $c = 2$ and $y = \cos x (\sin x + 2)$ M1

$x = \frac{\pi}{3}$: $y = \frac{1}{2}(\frac{\sqrt{3}}{2} + 2) = \boxed{\frac{\sqrt{3}}{4} + 1}$ A1

(Total for Question 4 is 8 marks)

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5.

$$f(x) = e^{\sin x}$$

(a) Show that $f''(x) = \left(\cos^2 x - \sin x\right) e^{\sin x}$

(3)

(b) Hence find the Maclaurin series expansion of $f(x)$, in ascending powers of x, up to and including the term in x^3

(3)

(c) Use your answer to part (b) to find an estimate for $\int_0^{0.2} e^{\sin x}\,dx$, giving your answer to 4 decimal places.

(2)

(a) $f'(x) = \cos x\, e^{\sin x}$ (chain rule) B1

$f''(x) = -\sin x\, e^{\sin x} + \cos x \times \cos x\, e^{\sin x}$ (product rule) M1

$= (\cos^2 x - \sin x)\, e^{\sin x}$ as required A1

(b) $f'''(x) = (-2 \sin x \cos x - \cos x)\, e^{\sin x} + (\cos^2 x - \sin x) \cos x\, e^{\sin x}$ M1

$f(0) = 1,\ f'(0) = 1,\ f''(0) = 1 - 0 = 1,\ f'''(0) = (0 - 1) + (1)(1) = 0$ A1

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$; $e^{\sin x} \approx 1 + x + \frac{1}{2}x^2 + 0x^3$ A1

(c) $\int_0^{0.2} (1 + x + \frac{1}{2}x^2)\,dx = [x + \frac{1}{2}x^2 + \frac{1}{6}x^3]_0^{0.2} = 0.2 + 0.02 + 0.00133\ldots$ M1

0.2213 A1

(Total for Question 5 is 8 marks)

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6. **(a)** On the same Argand diagram, sketch the locus of points representing complex numbers z satisfying

$$|z-3-4i| = 2 \qquad \text{and} \qquad \arg(z-1) = \frac{\pi}{4}$$

(3)

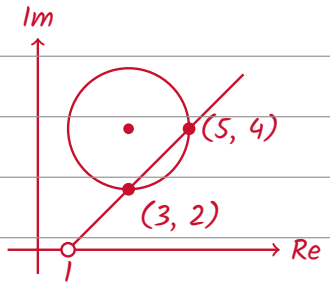
(b) Find the complex numbers z that satisfy both $|z-3-4i| = 2$ and $\arg(z-1) = \frac{\pi}{4}$

(3)

(c) Find the greatest value of $|z|$ for points on the locus $|z-3-4i| = 2$

(2)

(a) Circle: centre (3, 4), radius 2. Half-line: from (1, 0) at $\frac{\pi}{4}$ to the real axis B1 B1



B1

(b) Half-line: $y = x - 1$, $x > 1$. Circle: $(x-3)^2 + (y-4)^2 = 4$ M1

$(x-3)^2 + (x-5)^2 = 4 \Rightarrow 2x^2 - 16x + 30 = 0 \Rightarrow x^2 - 8x + 15 = 0 \Rightarrow x = 3$ or $x = 5$ A1

Both have $x > 1$, so both are on the half-line: $z = 3 + 2i$ and $z = 5 + 4i$ A1

(c) Greatest $|z|$ is on the line from 0 through the centre: $|3 + 4i| + 2 = 5 + 2$ M1

7 A1

(Total for Question 6 is 8 marks)

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7.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10\cos x$$

(a) Given that $y = \lambda\cos x + \mu\sin x$ is a particular integral of the differential equation, find the value of λ and the value of μ

(4)

(b) Find the general solution of the differential equation.

(2)

(c) Find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

(3)

(a) $y = \lambda\cos x + \mu\sin x$, $y' = -\lambda\sin x + \mu\cos x$, $y'' = -\lambda\cos x - \mu\sin x$ M1

$y'' + 2y' + 5y = (-\lambda + 2\mu + 5\lambda)\cos x + (-\mu - 2\lambda + 5\mu)\sin x = 10\cos x$ M1

$\cos x$: $4\lambda + 2\mu = 10$ $\quad$ $\sin x$: $4\mu - 2\lambda = 0$, so $\lambda = 2\mu$ A1

$10\mu = 10$: $\boxed{\mu = 1,\ \lambda = 2}$ A1

(b) Auxiliary equation: $m^2 + 2m + 5 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$ M1

$\boxed{y = e^{-x}(A\cos 2x + B\sin 2x) + 2\cos x + \sin x}$ A1

(c) $x = 0, y = 0$: $A + 2 = 0$, so $A = -2$ B1

$y' = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) - 2\sin x + \cos x$

$x = 0, y' = 0$: $-A + 2B + 1 = 0 \Rightarrow 2 + 2B + 1 = 0 \Rightarrow B = -\frac{3}{2}$ M1

$\boxed{y = e^{-x}(-2\cos 2x - \frac{3}{2}\sin 2x) + 2\cos x + \sin x}$ A1

(Total for Question 7 is 9 marks)

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8.

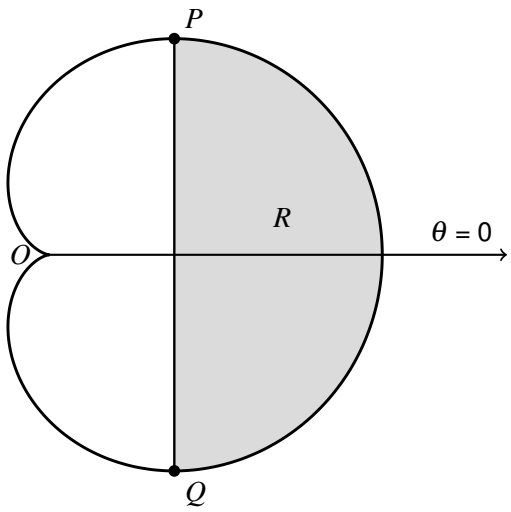

Figure 1

Figure 1 shows the curve C with polar equation

$$r = 4(1 + \cos\theta), \qquad -\pi < \theta \leqslant \pi$$

At the points P and Q on C the tangent to C is parallel to the initial line.

(a) Find the polar coordinates of P and the polar coordinates of Q

(5)

The finite region R, shown shaded in Figure 1, is bounded by C and the straight line PQ

(b) Use calculus to find the exact area of R, giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are integers.

(5)

(a) Need $dy/d\theta = 0$: $y = r\sin\theta = 4(1+\cos\theta)\sin\theta = 4\sin\theta + 2\sin 2\theta$ M1

$\frac{dy}{d\theta} = 4\cos\theta + 4\cos 2\theta = 0$ M1 A1

$4\cos\theta + 4(2\cos^2\theta - 1) = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$ M1

$\cos\theta = -1$ gives $r = 0$ (the pole), so $\cos\theta = \frac{1}{2}$: $\theta = \pm\frac{\pi}{3}$, $r = 4(1 + \frac{1}{2}) = 6$

$P(6, \frac{\pi}{3})$ $Q(6, -\frac{\pi}{3})$ A1

(b) Area of R = polar area from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$ minus triangle OPQ

$\frac{1}{2}\int 16(1+\cos\theta)^2\,d\theta = 8\int (1 + 2\cos\theta + \cos^2\theta)\,d\theta = 8\int (\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta)\,d\theta$ M1

$= 8[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta]_{-\pi/3}^{\pi/3} = 8(\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}) = 8\pi + 18\sqrt{3}$ M1 A1

$P(3, 3\sqrt{3})$, $Q(3, -3\sqrt{3})$: $OPQ = \frac{1}{2} \times 6\sqrt{3} \times 3 = 9\sqrt{3}$, so $R = 8\pi + 9\sqrt{3}$ M1 A1

(Total for Question 8 is 10 marks)

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9. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Use de Moivre's theorem to show that

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(6)

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{3}} \cos^5\theta \, d\theta$$

(4)

(c) Hence solve, for $0 \leqslant \theta \leqslant \pi$, the equation

$$\cos 5\theta + 5\cos 3\theta + 10\cos\theta = 0$$

(2)

(a) $z = \cos\theta + i\sin\theta$: $z^n + \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta$ B1

$(z + \frac{1}{z})^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$ (binomial: 1, 5, 10, 10, 5, 1) M1 A1

$= (z^5 + \frac{1}{z^5}) + 5(z^3 + \frac{1}{z^3}) + 10(z + \frac{1}{z})$ M1

$(2\cos\theta)^5 = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$, so $32\cos^5\theta = 2\cos 5\theta + \ldots$ A1

$\div 32$: $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$ as required A1

(b) $\int_0^{\pi/3} \cos^5\theta \, d\theta = \frac{1}{16}[\frac{1}{5}\sin 5\theta + \frac{5}{3}\sin 3\theta + 10\sin\theta]_0^{\pi/3}$ M1 A1

$\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$, $\sin\pi = 0$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$: $= \frac{1}{16}(-\frac{\sqrt{3}}{10} + 0 + 5\sqrt{3})$ M1

$= \frac{1}{16} \times \frac{49\sqrt{3}}{10} = \boxed{\frac{49\sqrt{3}}{160}}$ A1

(c) By part (a), the left side is $16\cos^5\theta$, so $16\cos^5\theta = 0 \Rightarrow \cos\theta = 0$ M1

$\boxed{\theta = \frac{\pi}{2}}$ (the only value in $0 \leqslant \theta \leqslant \pi$) A1

(Total for Question 9 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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