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A-LEVEL MATHEMATICS • FURTHER PURE 2

Complex Numbers

Worked Solutions

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Full worked solutions for the free Complex Numbers worksheet, A-Level and International A-Level Maths

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Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 2 Complex Numbers worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 2 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 $r = 2$; third quadrant: $-\pi + \frac{\pi}{6}$: $\boxed{2e^{-\frac{5\pi}{6}i}}$

A2 $2e^{(\frac{2\pi}{3}+\frac{\pi}{4})i}$: $\boxed{2e^{\frac{11\pi}{12}i}}$

A3 $\left(2e^{\frac{2\pi}{3}i}\right)^5 = 32e^{\frac{10\pi}{3}i} = 32e^{-\frac{2\pi}{3}i}$: $\boxed{-16-16\sqrt{3}i}$

STRETCH

A4 Imaginary part of $(c+is)^3$: $3c^2s - s^3 = 3(1-s^2)s - s^3$: $\boxed{3\sin\theta - 4\sin^3\theta}$

A5 $\left(z+\frac{1}{z}\right)^5 = \left(z^5+z^{-5}\right) + 5\left(z^3+z^{-3}\right) + 10\left(z+z^{-1}\right)$, so $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$:

$\boxed{\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)}$

A6 $27e^{(\frac{\pi}{2}+2k\pi)i}$, so $3e^{(\frac{\pi}{6}+\frac{2k\pi}{3})i}$: $\boxed{3e^{\frac{\pi}{6}i},\ 3e^{\frac{5\pi}{6}i},\ 3e^{-\frac{\pi}{2}i}}$

A7 Vertices on a circle of radius 2; side $2\sqrt{2}$: $\boxed{8}$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** $\boxed{3e^{\frac{\pi}{2}i}}$ **(b)** $\boxed{5e^{\pi i}}$ **(c)** $\boxed{2e^{-\frac{\pi}{3}i}}$ **(d)** $\boxed{3\sqrt{2}e^{-\frac{3\pi}{4}i}}$

Check the quadrant every time.

B2 **(a)** $\boxed{8e^{-\frac{\pi}{12}i}}$ **(b)** $\boxed{\frac{1}{2}e^{\frac{5\pi}{12}i}}$ **(c)** $\boxed{64e^{\pi i} = -64}$ **(d)** $\boxed{64e^{-\frac{3\pi}{4}i}}$

Moduli multiply or are raised to the power; arguments add or are multiplied.

B3 **(a)** $(\sqrt{2})^8 e^{2\pi i}$: $\boxed{16}$ **(b)** $64e^{-\pi i}$: $\boxed{-64}$ **(c)** $\frac{1}{8}e^{-\pi i}$: $\boxed{-\frac{1}{8}}$ **(d)** $\frac{n\pi}{4} = 2\pi$: $\boxed{8}$

Negative powers work too.

B4 **(a)** $\boxed{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$ **(b)** $\boxed{8\sin^4\theta - 8\sin^2\theta + 1}$

(c) $\boxed{16\cos^4\theta - 12\cos^2\theta + 1}$ **(d)** $\boxed{3 - 4\sin^2 2\theta}$

Real parts give cosines; imaginary parts give sines.

B5 **(a)** $z^n + z^{-n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta$ **(b)** similarly $2i\sin n\theta$

(c) $(2\cos\theta)^2(2i\sin\theta)^2 = \left(z^2 - z^{-2}\right)^2 = 2\cos 4\theta - 2$, so $-16\sin^2\theta\cos^2\theta = 2\cos 4\theta - 2$, as required

These two results turn powers into multiple angles.

B6 **(a)** $\boxed{2, 2e^{\pm\frac{2\pi}{3}i}}$ **(b)** $\boxed{\pm 3, 3e^{\pm\frac{\pi}{2}i}}$ **(c)** $\boxed{e^{\pm\frac{\pi}{6}i}, e^{\pm\frac{\pi}{2}i}, e^{\pm\frac{5\pi}{6}i}}$ **(d)** $\boxed{\pm(1+i)}$

n roots, equally spaced by $\frac{2\pi}{n}$

STRETCH

B7 **(a)** $\cos 4\alpha = 0$: $\boxed{\cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}}$ **(b)** $u^2 - u + \frac{1}{8} = 0$: $\boxed{\frac{2+\sqrt{2}}{4}}$

(c) $\boxed{\frac{1}{8}}$ **(d)** $\left(\cos\frac{\pi}{8}\cos\frac{3\pi}{8}\right)^2 = \frac{1}{8}$: $\boxed{\frac{\sqrt{2}}{4}}$

The identity turns the quartic into $\cos 4\alpha = 0$

B8 **(a)** $\boxed{\frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)}$ **(b)** $\boxed{\frac{3\pi}{16}}$ **(c)** $\boxed{\frac{1}{4}(\cos 3\theta + 3\cos\theta)}$ **(d)** $\left[\frac{\sin 3\theta}{12} + \frac{3}{4}\sin\theta\right]_0^{\frac{\pi}{3}}$: $\boxed{\frac{3\sqrt{3}}{8}}$

Multiple angles integrate directly.

B9 **(a)** $\boxed{2e^{\frac{k\pi}{3}i},\ k = -2, \ldots, 3}$ **(b)** $\boxed{0}$ **(c)** $6 \times \frac{1}{2} \times 2^2 \sin\frac{\pi}{3}$: $\boxed{6\sqrt{3}}$ **(d)** $\boxed{-64}$

The roots of $z^6 - 64$: the sum is minus the z^5 coefficient, the product is the constant term (with sign $(-1)^6$)

B10 **(a)** $\boxed{8e^{\frac{3\pi}{4}i}}$ **(b)** $\boxed{2e^{\frac{\pi}{4}i}, 2e^{\frac{11\pi}{12}i}, 2e^{-\frac{5\pi}{12}i}}$ **(c)** $\boxed{\sqrt{2} + \sqrt{2}i}$ **(d)** $(\sqrt{2} + \sqrt{2}i)^3 = -4\sqrt{2} + 4\sqrt{2}i$

A root in $a + bi$ form can be checked by expanding.

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C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 $-1-\mathrm{i}$ is in the third quadrant: $\boxed{-\frac{3\pi}{4}}$

C2 The modulus must also be raised to the power 4: $16\mathrm{e}^{\frac{4\pi}{3}\mathrm{i}}$: $\boxed{16\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}}}$

C3 He omitted the binomial coefficient 3: $c^3 - 3cs^2 = c^3 - 3c(1-c^2)$: $\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$

C4 There are four fourth roots: $\boxed{3\mathrm{e}^{\pm\frac{\pi}{4}\mathrm{i}},\ 3\mathrm{e}^{\pm\frac{3\pi}{4}\mathrm{i}}}$

STRETCH

C5 She did not square-root: $\sin^2\theta = \frac{1}{4}$, $\sin\theta = \frac{1}{2}$: $\boxed{\theta = \frac{\pi}{6}}$

C6 $\cos\frac{\pi}{8}$ is the largest root; the smallest positive is $\cos\frac{3\pi}{8}$: $\boxed{\mathbf{0.383}}$

C7 $\frac{7\pi}{6} > \pi$; subtract 2π: $\boxed{2\mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}}}$

C8 **(a)** $n = 1$ is true. If $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$, multiplying by $\cos\theta + \mathrm{i}\sin\theta$ and using the compound-angle formulae gives $\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$; so true for all positive integers n

(b) For $n = -m$: $(\cos\theta + \mathrm{i}\sin\theta)^{-m} = \dfrac{1}{\cos m\theta + \mathrm{i}\sin m\theta} = \cos m\theta - \mathrm{i}\sin m\theta = \cos n\theta + \mathrm{i}\sin n\theta$, as required

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D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 **(a)** $r = 6$; second quadrant: $\boxed{6e^{\frac{5\pi}{6}i}}$

(b) $1296e^{\frac{10\pi}{3}i} = 1296e^{-\frac{2\pi}{3}i}$: $\boxed{-648 - 648\sqrt{3}i}$

D2 $z^4 = -8 - 8\sqrt{3}i = 16e^{-\frac{2\pi}{3}i}$, so $z = 2e^{\left(-\frac{\pi}{6}+\frac{k\pi}{2}\right)i}$: $\boxed{2e^{-\frac{\pi}{6}i},\ 2e^{\frac{\pi}{3}i},\ 2e^{\frac{5\pi}{6}i},\ 2e^{-\frac{2\pi}{3}i}}$

D3 **(a)** $\boxed{25}$

(b) $(a+ib)^2 = -7+24i$: $a^2 - b^2 = -7$, $ab = 12$: $\boxed{w = \pm(3+4i)}$

D4 $\dfrac{e^{\frac{6\pi}{9}i}}{e^{-\frac{2\pi}{6}i}} = e^{\left(\frac{2\pi}{3}+\frac{\pi}{3}\right)i} = e^{\pi i} = -1$, as required

STRETCH

D5 **(a)** $\cos 4\theta = c^4 - 6c^2s^2 + s^4$, $\sin 4\theta = 4c^3s - 4cs^3$; divide by c^4, giving the result

(b) $\tan 4\theta = 1$: $\theta = \dfrac{\pi}{16} + \dfrac{k\pi}{4}$: $\boxed{0.199,\ 1.497,\ -5.027,\ -0.668}$

D6 **(a)** $\left(z - z^{-1}\right)^5 = \left(z^5 - z^{-5}\right) - 5\left(z^3 - z^{-3}\right) + 10\left(z - z^{-1}\right)$: $32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$, as required

(b) $\dfrac{1}{16}\left[-\dfrac{\cos 5\theta}{5} + \dfrac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}} = \dfrac{1}{16}\left(\dfrac{1}{5} - \dfrac{5}{3} + 10\right)$: $\boxed{\dfrac{8}{15}}$

D7 **(a)** $32u^3 - 48u^2 + 18u - 1 = (2u-1)(16u^2 - 16u + 1)$, as required

(b) $u = \dfrac{1}{2}$ gives $\theta = \dfrac{\pi}{4}$; $16u^2 - 16u + 1 = 0$ gives $u = \dfrac{2 \pm \sqrt{3}}{4}$: $\boxed{\cos^2\dfrac{\pi}{12} = \dfrac{2+\sqrt{3}}{4},\ \cos^2\dfrac{5\pi}{12} = \dfrac{2-\sqrt{3}}{4}}$

D8 **(a)** $\boxed{8e^{\frac{\pi}{6}i}}$

(b) $z = 2e^{\left(\frac{\pi}{18}+\frac{2k\pi}{3}\right)i}$: $\boxed{2e^{\frac{\pi}{18}i},\ 2e^{\frac{13\pi}{18}i},\ 2e^{-\frac{11\pi}{18}i}}$

(c) An equilateral triangle in a circle of radius 2: $3 \times \dfrac{1}{2} \times 2^2 \sin\dfrac{2\pi}{3}$: $\boxed{3\sqrt{3}}$

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 **(a)** $r = \sqrt{768 + 256} = 32$ *B1*

third quadrant: $\boxed{32e^{-\frac{5\pi}{6}i}}$ *B1*

(b) $z = 2e^{\left(-\frac{\pi}{6} + \frac{2k\pi}{5}\right)i}$ *M1 A1*

$\boxed{2e^{-\frac{\pi}{6}i},\ 2e^{\frac{7\pi}{30}i},\ 2e^{\frac{19\pi}{30}i},\ 2e^{-\frac{17\pi}{30}i},\ 2e^{-\frac{29\pi}{30}i}}$ *M1 A1*

E2 **(a)** $\boxed{z = \sqrt{2}e^{\frac{\pi}{4}i},\ w = 2e^{-\frac{\pi}{6}i}}$ *B1 B1*

(b) $\boxed{\frac{\sqrt{2}}{2}e^{\frac{5\pi}{12}i}}$ *B1*

(c) $z^8 = 16e^{2\pi i} = 16,\ w^6 = 64e^{-\pi i} = -64$ *M1*

$\boxed{-1024}$ *A1*

(d) $\frac{5n\pi}{12}$ must be a multiple of π *M1*

$\boxed{n = 12}$ *A1*

STRETCH

E3 **(a)** $(c + is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3$ *M1*

$= c^3 - 3cs^2 + i\left(3c^2s - s^3\right)$ *A1*

equating real and imaginary parts with $\cos 3\theta + i\sin 3\theta$, as required *A1**

(b) $\tan 3\theta = \frac{3c^2s - s^3}{c^3 - 3cs^2}$ *M1*

dividing by c^3 gives the result *A1**

(c) $t^3 - 3t^2 - 3t + 1 = 0 \iff 3t - t^3 = 1 - 3t^2 \iff \tan 3\theta = 1$ *M1*

$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, so $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$ *M1 A1*

$\boxed{t = 2 - \sqrt{3},\ 2 + \sqrt{3},\ -1}$ *A1*