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A-LEVEL MATHEMATICS • FURTHER PURE 3

Hyperbolic Functions

Worked Solutions

Pearson Edexcel International A Level (WFM03) • Calculator allowed: show every step

Full worked solutions for the free Hyperbolic Functions worksheet, A-Level and International A-Level Maths

A **Examples** & Your turn page 2

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Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 3 Hyperbolic Functions worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 3 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 $\frac{1}{2}\left(4+\frac{1}{4}\right)$ and $\frac{16-1}{16+1}$: $\boxed{\frac{17}{8}, \frac{15}{17}}$

A2 $\frac{e^{2x}+2+e^{-2x}}{4}+\frac{e^{2x}-2+e^{-2x}}{4}=\frac{e^{2x}+e^{-2x}}{2}=\cosh 2x$, as required

A3 $\frac{1}{2}\ln\frac{1+\frac{3}{5}}{1-\frac{3}{5}}=\frac{1}{2}\ln 4$: $\boxed{\ln 2}$

STRETCH

A4 $e^{x}+2e^{-x}=3$, $u^{2}-3u+2=0$, $u=1,2$: $\boxed{x=0, \ln 2}$

A5 $\sinh x=3$ or -1 (both allowed): $\boxed{x=\ln\left(3+\sqrt{10}\right), \ln\left(\sqrt{2}-1\right)}$

A6 $\cosh x=\frac{5}{3}$: $\boxed{x=\pm\ln 3}$

A7 $R^{2}=169-25=144$, $\tanh\alpha=\frac{5}{13}$, $\alpha=\frac{1}{2}\ln\frac{9}{4}$: $\boxed{12\cosh\left(x+\ln\frac{3}{2}\right)}$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** $\boxed{\frac{12}{5}}$ **(b)** $\boxed{\frac{13}{5}}$ **(c)** $\boxed{\frac{12}{13}}$ **(d)** $\boxed{\frac{5}{13}}$

$e^{\ln 5}=5$*; cosh and sech are even.*

B2 **(a)** $\boxed{\ln\left(1+\sqrt{2}\right)}$ **(b)** $\boxed{\ln\left(3+2\sqrt{2}\right)}$ **(c)** $\boxed{\frac{1}{2}\ln 2}$ **(d)** $\boxed{\ln\left(\sqrt{5}-2\right)}$

The formula-book forms give exact logarithms.

B3 **(a)** $\boxed{\cosh 2x \equiv 2\cosh^2 x - 1}$ **(b)** $\boxed{\sinh 3x \equiv 3\sinh x + 4\sinh^3 x}$ **(c)** $\boxed{\cosh 3x \equiv 4\cosh^3 x - 3\cosh x}$

(d) $\boxed{\sinh^2 x \equiv \frac{1}{2}(\cosh 2x - 1)}$

Change the sign of every product of two sines.

B4 **(a)** $\boxed{\ln\left(2+\sqrt{5}\right)}$ **(b)** $\boxed{\pm\ln\left(2+\sqrt{3}\right)}$ **(c)** $3u^2 - 3u - 1 = 0$: $\boxed{\ln\frac{3+\sqrt{21}}{6}}$ **(d)** $e^{-x} = 2$: $\boxed{-\ln 2}$

Reject the negative value of e^x

B5 **(a)** $\boxed{\textbf{all real numbers}}$ **(b)** $\boxed{1 < y < 7}$ **(c)** it gives $e^{-x} = 0$, which is impossible

The graphs decide which values are possible.

B6 **(a)** $\cosh x = 2$: $\boxed{\pm\ln\left(2+\sqrt{3}\right)}$ **(b)** $\sinh x = \frac{1}{2}$ or -3: $\boxed{\ln\frac{1+\sqrt{5}}{2}, \ln\left(\sqrt{10}-3\right)}$

(c) $\cosh^2 x = 9$, $\cosh x = 3$: $\boxed{\pm\ln\left(3+2\sqrt{2}\right)}$ **(d)** $\tanh x = \pm\frac{1}{2}$: $\boxed{\pm\frac{1}{2}\ln 3}$

Reduce to one function, then check every value is possible.

STRETCH

B7 **(a)** , **(b)** add or subtract the definitions **(c)** $(e^x)^n = e^{nx} = \cosh nx + \sinh nx$ **(d)** $e^{3x} = 8$: $\boxed{x = \ln 2}$

$\cosh x + \sinh x$ *is just* e^x

B8 **(a)** divide $\cosh^2 x - \sinh^2 x \equiv 1$ by $\sinh^2 x$ **(b)** $\boxed{\operatorname{cosech}^2 x - 3\operatorname{cosech} x - 4 = 0}$ **(c)** $\boxed{\textbf{4 or } -1}$

(d) $\sinh x = \frac{1}{4}$ or -1: $\boxed{\ln\frac{1+\sqrt{17}}{4}, \ln\left(\sqrt{2}-1\right)}$

Both values are allowed: cosech takes every non-zero value.

B9 **(a)** $R^2 = 289 - 64$: $\boxed{15}$ **(b)** $\tanh\alpha = \frac{8}{17}$: $\boxed{\ln\frac{5}{3}}$ **(c)** $\boxed{15}$ **(d)** $\cosh(x+\alpha) = \frac{17}{15}$: $\boxed{x = 0, -2\ln\frac{5}{3}}$

The least value of $R\cosh(x+\alpha)$ *is* R

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B10 **(a)** $\boxed{x \geqslant 1}$ **(b)** $\boxed{-1 < x < 1}$ **(c)** $\boxed{\frac{1}{2}\ln 3}$ **(d)** $\frac{1+x}{1-x} = 4$: $\boxed{x = \frac{3}{5}}$

The domains come from the ranges of cosh and tanh.

C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 The question asks for the exponential definitions to be used; Osborn's rule is not a proof.

C2 $\cosh x \geqslant 1$ for all x, so $\cosh x = -2$ has **no solutions**

C3 He has lost the $\frac{1}{2}$: $e^x + e^{-x} = 5$, $u^2 - 5u + 1 = 0$: $\boxed{x = \pm\ln\frac{5+\sqrt{21}}{2}}$

C4 $e^x > 0$, so $e^x = -5$ is impossible: $\boxed{x = \ln 3}$ **only**

STRETCH

C5 cosh is even, so there are two solutions: $\boxed{x = \pm\ln\left(5+2\sqrt{6}\right)}$

C6 She must square-root: $\cosh x = 2$: $\boxed{x = \pm\ln\left(2+\sqrt{3}\right)}$

C7 $-1 < \tanh x < 1$ for all x, so $\tanh x = 2$ is impossible (the logarithm of a negative number confirms it).

C8 **(a)** $x = \frac{e^{2y}-1}{e^{2y}+1}$, so $e^{2y} = \frac{1+x}{1-x}$ and $y = \frac{1}{2}\ln\frac{1+x}{1-x}$, as required

(b) $\tanh y$ only takes values in $(-1, 1)$; and $\frac{1+x}{1-x}$ must be positive

D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 $2\cosh^2 x - 2 = 5\cosh x - 5$, $2\cosh^2 x - 5\cosh x + 3 = 0$, $\cosh x = 1$ or $\frac{3}{2}$: $\boxed{x = 0, \pm\ln\frac{3+\sqrt{5}}{2}}$

D2 **(a)** $\frac{1}{4}\left[\left(e^A + e^{-A}\right)\left(e^B + e^{-B}\right) - \left(e^A - e^{-A}\right)\left(e^B - e^{-B}\right)\right] = \frac{1}{2}\left(e^{A-B} + e^{B-A}\right)$, as required

(b) $\frac{5}{3}\cosh x - \frac{4}{3}\sinh x = 3\sinh x$, $\tanh x = \frac{5}{13}$: $\boxed{x = \ln\frac{3}{2}}$

D3 $2e^x + 4e^{-x} = 3e^x - 1$, $u^2 - u - 4 = 0$, $u = \frac{1+\sqrt{17}}{2}$ (reject negative): $\boxed{x = \ln\frac{1+\sqrt{17}}{2}}$

D4 $\frac{1}{2}e^x + \frac{5}{2}e^{-x} = 2$, $u^2 - 4u + 5 = 0$, discriminant $16 - 20 < 0$, so **no real solutions**

STRETCH

D5 **(a)** $R\cosh\alpha = 13$, $R\sinh\alpha = 5$: $\boxed{12\sinh\left(x + \ln\frac{3}{2}\right)}$

(b) $\sinh\left(x + \ln\frac{3}{2}\right) = 1$, $x + \ln\frac{3}{2} = \ln\left(1+\sqrt{2}\right)$: $\boxed{x = \ln\frac{2\left(1+\sqrt{2}\right)}{3}}$

D6 **(a)** $4 \times \frac{1}{8}\left(e^x + e^{-x}\right)^3 = \frac{1}{2}\left(e^{3x} + 3e^x + 3e^{-x} + e^{-3x}\right) = \cosh 3x + 3\cosh x$, as required

(b) $4\cosh^3 x - 3\cosh x = 9\cosh x$, $\cosh^2 x = 3$: $\boxed{x = \pm\ln\left(\sqrt{3}+\sqrt{2}\right)}$

D7 $\frac{u^2-1}{u^2+1} + \frac{6u}{u^2+1} = 3$, $u^2 - 3u + 2 = 0$: $\boxed{x = 0, \ln 2}$

D8 **(a)** $1 + 2 \times \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$, as required

(b) $2\sinh^2 x - 7\sinh x + 3 = 0$, $\sinh x = 3$ or $\frac{1}{2}$: $\boxed{x = \ln\left(3+\sqrt{10}\right), \ln\frac{1+\sqrt{5}}{2}}$

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 $3\left(1+\sinh^2 x\right)-4\sinh x=7$ *M1*

$3\sinh^2 x-4\sinh x-4=0$ *A1*

$(3\sinh x+2)(\sinh x-2)=0$, $\sinh x=2$ or $-\frac{2}{3}$ *M1 A1*

$x=\ln\left(2+\sqrt{5}\right)$ *A1*

$\boxed{x=\ln\left(2+\sqrt{5}\right),\ x=\ln\frac{\sqrt{13}-2}{3}}$ *A1*

E2 **(a)** $\frac{1}{4}\left[\left(e^{A}-e^{-A}\right)\left(e^{B}+e^{-B}\right)-\left(e^{A}+e^{-A}\right)\left(e^{B}-e^{-B}\right)\right]$ *M1*

$=\frac{1}{2}\left(e^{A-B}-e^{B-A}\right)=\sinh(A-B)$, as required *A1**

(b) $R\cosh\alpha=5$, $R\sinh\alpha=3$ *M1*

$R=4$, $\tanh\alpha=\frac{3}{5}$ *A1*

$\boxed{4\sinh(x-\ln 2)}$ *A1*

(c) $\sinh(x-\ln 2)=1$, $x-\ln 2=\ln\left(1+\sqrt{2}\right)$ *M1*

$\boxed{x=\ln\left(2+2\sqrt{2}\right)}$ *A1*

STRETCH

E3 **(a)** $4\times\frac{1}{8}\left(e^{x}-e^{-x}\right)^3=\frac{1}{2}\left(e^{3x}-3e^{x}+3e^{-x}-e^{-3x}\right)$ *M1 A1*

$=\sinh 3x-3\sinh x$, as required *A1**

(b) $4\sinh^3 x+3\sinh x=19\sinh x$, so $\sinh x\left(\sinh^2 x-4\right)=0$ *M1*

$\sinh x=0$ or ± 2 *A1*

$\boxed{x=0,\ \pm\ln\left(2+\sqrt{5}\right)}$ *A1*

(c) $\cosh\beta=\sqrt{1+4}=\sqrt{5}$ *B1*

$\cosh 3\beta=4\cosh^3\beta-3\cosh\beta=20\sqrt{5}-3\sqrt{5}$ *M1*

$\boxed{17\sqrt{5}}$ *A1*

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