Proof

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 4

Question 1 — Worked Solution

Assume the negation: assume there exists an integer *n* such that n^2 is even but *n* is odd.

Since n is odd, write $n = 2k + 1$ for some integer k.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

This is of the form $2m + 1$ (odd), so n^2 is **odd**.

But we assumed n^2 is even.

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■

Question 2 — Worked Solution

Assume the negation: assume √2 is rational.

Then we can write $\sqrt{2} = p/q$ where p, q are integers with no common factors (i.e. the fraction is in its lowest terms).

$$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$$

So p^2 is even, which by Q1 means p is even. Write $p = 2m$.

$$(2m)^2 = 2q^2 \Rightarrow 4m^2 = 2q^2 \Rightarrow q^2 = 2m^2$$

So q^2 is even, meaning q is also even.

But then p and q are both even, contradicting the assumption that p/q is in lowest terms.

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■

Question 3 — Worked Solution

Assume the negation: assume $\log_2 3$ is rational.

Then $\log_2 3 = p/q$ for some positive integers p, q.

$$2^{p/q} = 3 \Rightarrow 2^p = 3^q$$

The left-hand side 2^p is a power of 2, so it is **even**.

The right-hand side 3^q is a power of 3, so it is **odd**.

An even number cannot equal an odd number.

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■

Question 4 — Worked Solution

Assume the negation: assume there exists a real number x such that $3x^2 + 7 = 0$.

$$3x^2 = -7 \;\Rightarrow\; x^2 = -\frac{7}{3}$$

But $x^2 \geq 0$ for all real x, so x^2 cannot equal $-7/3 < 0$.

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■

Question 5 — Worked Solution

Assume the negation: assume there exists a greatest even integer *N*.

Consider $N + 2$.

$N + 2$ is even (sum of two even numbers).

$N + 2 > N$, so $N + 2$ is a greater even integer than N.

This contradicts the assumption that N is the greatest even integer.

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■

Question 6 — Worked Solution

Assume the negation: assume √2 + √3 is rational.

Then √2 + √3 = *r* for some rational number *r*. Note $r \neq 0$ since √2 + √3 > 0.

$$\sqrt{3} = r - \sqrt{2}$$

Square both sides:

$$3 = r^2 - 2r\sqrt{2} + 2$$

Rearrange:

$$2r\sqrt{2} = r^2 - 1 \;\Rightarrow\; \sqrt{2} = \frac{r^2 - 1}{2r}$$

Since *r* is rational, $(r^2-1)/(2r)$ is rational, which would make √2 rational.

But √2 is irrational (given).

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■

Question 7 — Worked Solution

Assume the negation: assume there exist a rational number *a* and an irrational number *b* such that *a* + *b* is rational.

Let $a + b = c$ where c is rational.

$$b = c - a$$

Since c and a are both rational, their difference $c - a$ is rational.

Therefore b is rational.

But b was assumed to be irrational.

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■

Question 8 — Worked Solution

Assume the negation: assume there are only finitely many prime numbers, say $p_1, p_2, \ldots, p_n$.

Define the integer:

$$N = p_1 \times p_2 \times \cdots \times p_n + 1$$

$N > 1$ so N must have at least one prime factor, call it q.

Now q must be one of our listed primes $p_1, \ldots, p_n$ (since we assumed these are all the primes).

But each p_i divides $p_1 \times \ldots \times p_n$, so each p_i leaves a remainder of 1 when dividing N.

Therefore q does not divide N — yet q was defined as a prime factor of N.

This is a contradiction. Therefore our assumption was false, and the original statement is true. ■