Differentiation

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 3

Question 1 — Worked Solution

(a) $y = \sin 3x \Rightarrow \dfrac{dy}{dx} = 3\cos 3x$

(b) $y = \cos(x^2) \Rightarrow \dfrac{dy}{dx} = -\sin(x^2) \cdot 2x = -2x\sin(x^2)$

(c) $y = e^{4x} \Rightarrow \dfrac{dy}{dx} = 4e^{4x}$

(d) $y = \ln(2x+1) \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{2x+1}$

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Question 2 — Worked Solution

(a)

$$y = (3x^2 + 1)^5, \quad \frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot 6x = 30x(3x^2 + 1)^4$$

(b)

$$y = (x^3 - 2x)^{\frac{1}{2}}, \quad \frac{dy}{dx} = \frac{1}{2}(x^3 - 2x)^{-\frac{1}{2}} \cdot (3x^2 - 2) = \frac{3x^2 - 2}{2\sqrt{x^3 - 2x}}$$

(c)

$$y = e^{\sin x}, \quad \frac{dy}{dx} = e^{\sin x} \cdot \cos x$$

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Question 3 — Worked Solution

(a) y = x^2 sin x: u = x^2, v = sin x

$$\frac{dy}{dx} = 2x\sin x + x^2\cos x = x(2\sin x + x\cos x)$$

(b) y = e^x cos x: u = e^x, v = cos x

$$\frac{dy}{dx} = e^x\cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$$

(c) y = x^3 ln x: u = x^3, v = ln x

$$\frac{dy}{dx} = 3x^2\ln x + x^3 \cdot \frac{1}{x} = 3x^2\ln x + x^2 = x^2(3\ln x + 1)$$

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Question 4 — Worked Solution

(a) y = sinx/x: u = sin x, v = x

$$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$$

(b) y = e^x/(x^2+1):   u = e^x, v = x^2+1

$$\frac{dy}{dx} = \frac{e^x(x^2+1) - e^x(2x)}{(x^2+1)^2} = \frac{e^x(x^2-2x+1)}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2}$$

(c) y = ln x / cos x: u = ln x, v = cos x

$$\frac{dy}{dx} = \frac{\frac{1}{x}\cos x - \ln x(-\sin x)}{\cos^2 x} = \frac{\cos x}{x\cos^2 x} + \frac{\ln x \sin x}{\cos^2 x}$$

$$= \frac{\sec x}{x} + \ln x \sec x \tan x$$

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Question 5 — Worked Solution

(a) y = ln(sin x)

Chain rule: outer = ln, inner = sin x

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$\therefore = \cot x$$

(b) y = sin(ln x)

Chain rule: outer = sin, inner = ln x

$$\frac{dy}{dx} = \cos(\ln x) \cdot \frac{1}{x} = \frac{\cos(\ln x)}{x}$$

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Question 6 — Worked Solution

$y = xe^x$

$$\frac{dy}{dx} = e^x + xe^x = e^x(1+x)$$

At x=1: y = e, dy/dx = 2e

(a) Tangent

$$y - e = 2e(x-1) \Rightarrow y = 2ex - 2e + e = 2ex - e$$

$$\therefore y = e(2x-1)$$

(b) Normal

Normal gradient = –1/(2e).

$$y - e = -\frac{1}{2e}(x-1) \Rightarrow y = -\frac{x}{2e} + \frac{1}{2e} + e$$

$$\therefore y = -\frac{x}{2e} + e + \frac{1}{2e}$$

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Question 7 — Worked Solution

(a) Show dy/dx = $xe^{-x}(2-x)$

Product rule with $u=x^2$, $v=e^{-x}$:

$$\frac{dy}{dx} = 2xe^{-x} + x^2(-e^{-x}) = e^{-x}(2x - x^2) = xe^{-x}(2-x) \quad ✓$$

(b) Stationary points

$$xe^{-x}(2-x) = 0 \Rightarrow x = 0 \text{ or } x = 2$$

$$x = 0: \ y = 0 \Rightarrow (0,\ 0)$$

$$x = 2: \ y = 4e^{-2} \Rightarrow (2,\ 4e^{-2})$$

Second derivative test or sign of dy/dx:

x=0: dy/dx changes – to + $\rightarrow$ **minimum**

x=2: dy/dx changes + to – $\rightarrow$ **maximum**

$$\therefore \text{Min at } (0,\ 0); \quad \text{Max at } (2,\ 4e^{-2})$$

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Question 8 — Worked Solution

(a) dy/dx

Product rule with u=x, v=ln x:

$$\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

(b) Stationary point

$$\ln x + 1 = 0 \;\Rightarrow\; \ln x = -1 \;\Rightarrow\; x = e^{-1} = \frac{1}{e}$$

$$y = \frac{1}{e}\ln\left(\frac{1}{e}\right) = \frac{1}{e} \cdot (-1) = -\frac{1}{e}$$

$$\therefore \left(\frac{1}{e},\ -\frac{1}{e}\right)$$

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Question 9 — Worked Solution

Find tangent to y = e^{2x}/(x+1) at x=0

Quotient rule with u=e^{2x}, v=x+1:

$$\frac{dy}{dx} = \frac{2e^{2x}(x+1) - e^{2x}}{(x+1)^2} = \frac{e^{2x}(2x+1)}{(x+1)^2}$$

At x=0: y = e^0/1 = 1, dy/dx = e^0(1)/1 = 1

$$y - 1 = 1(x - 0)$$

$$\therefore y = x + 1$$

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Question 10 — Worked Solution

Prove d/dx(tan x) = sec^{2}x

Apply quotient rule to y = sinx/cosx with u=sinx, v=cosx:

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Use sin^{2}x + cos^{2}x = 1:

$$= \frac{1}{\cos^2 x} = \sec^2 x \quad ✓$$

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Question 11 — Worked Solution

Differentiate $y = e^{x^2} \ln(\cos x)$

Product rule with $u=e^{x^2}$, $v=\ln(\cos x)$:

$$\frac{du}{dx} = 2xe^{x^2} \qquad \frac{dv}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$\frac{dy}{dx} = 2xe^{x^2}\ln(\cos x) + e^{x^2}(-\tan x)$$

$$= e^{x^2}[2x\ln(\cos x) - \tan x]$$

$$\therefore \frac{dy}{dx} = e^{x^2}[2x\ln(\cos x) - \tan x]$$

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Question 12 — Worked Solution

(a) Find dy/dx

Product rule with $u=x^2$, $v=\sin(3x)$:

$$\frac{dy}{dx} = 2x\sin(3x) + x^2 \cdot 3\cos(3x) = 2x\sin(3x) + 3x^2\cos(3x)$$

(b) Show d^2y/dx^2 = 2sin3x + 12xcos3x – $9x^2$sin3x

Differentiate dy/dx = 2xsin(3x) + $3x^2$cos(3x) using product rule on each term:

$$\frac{d}{dx}[2x\sin 3x] = 2\sin 3x + 2x \cdot 3\cos 3x = 2\sin 3x + 6x\cos 3x$$

$$\frac{d}{dx}[3x^2\cos 3x] = 6x\cos 3x + 3x^2(-3\sin 3x) = 6x\cos 3x - 9x^2\sin 3x$$

Add:

$$\frac{d^2y}{dx^2} = 2\sin 3x + 6x\cos 3x + 6x\cos 3x - 9x^2\sin 3x$$

$$= 2\sin 3x + 12x\cos 3x - 9x^2\sin 3x \quad ✓$$

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Question 13 — Worked Solution

(a) Show y = 1 – cos x

Use $\sin^2 x = 1 - \cos^2 x = (1-\cos x)(1+\cos x)$:

$$y = \frac{\sin^2 x}{1+\cos x} = \frac{(1-\cos x)(1+\cos x)}{1+\cos x} = 1 - \cos x \quad ✓$$

(b) dy/dx

$$\frac{dy}{dx} = \sin x$$

(c) Stationary points in $0 \le x \le 2\pi$

$$\sin x = 0 \;\Rightarrow\; x = 0,\ \pi,\ 2\pi$$

$$\therefore x = 0, \quad x = \pi, \quad x = 2\pi$$

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