eClassroom

A-LEVEL MATHEMATICS • MECHANICS 2

Centres of Mass

Worksheet

Pearson Edexcel International A Level (WME02) • Calculator allowed: take $g = 9.8\,\text{m}\,\text{s}^{-2}$ and give answers to 2 or 3 s.f.

Free Centres of Mass worksheet for A-Level and International A-Level Maths, with full worked solutions

A — **Examples** & Your turn — page 2

B — **Fluency** core skills — page 4

C — **Reasoning** explain it — page 6

D — **Problem** solving — page 7

E — **Exam** practice — page 10

Tap a section to jump straight to it

By the end of this worksheet you can: find the centre of mass of particles, laminas and frameworks • use a table of masses for composite shapes, with removed parts negative • find the angle at which a body hangs • find when a body is on the point of toppling • *Stretch:* different densities, folded laminas, held bodies and general results.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Holes, folds, wires, different densities, extra forces and toppling on a slope.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	19
Stretch marks	16	14	17	12
My score				

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A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: particles

Particles of mass 2 kg, 3 kg and 5 kg are at the points $(1,4)$, $(3,-2)$ and $(6,1)$. Find the coordinates of their centre of mass.

$$\bar{x} = \frac{\sum mx}{\sum m},\ \bar{y} = \frac{\sum my}{\sum m}$$

$$\bar{x} = \frac{2+9+30}{10};\ \bar{y} = \frac{8-6+5}{10}$$

$$(4.1,\ 0.7)$$

Your turn A1

Particles of mass 4 kg, 1 kg and 3 kg are at the points $(2,0)$, $(-1,5)$ and $(5,3)$. Find the coordinates of their centre of mass.

Worked example A2: a composite lamina

A uniform T-shaped lamina is made of a rectangle 10 cm by 2 cm on top of a central stem 2 cm wide and 8 cm tall. Find the height of its centre of mass above the bottom of the stem.

Mass $\propto$ area. Top: 20 at height 9; stem: 16 at height 4

$$\bar{y} = \frac{20(9)+16(4)}{36} = \frac{61}{9}$$

$$6.78\,\text{cm}$$

Your turn A2

A uniform lamina is a rectangle 6 cm wide and 4 cm tall with an isosceles triangle of base 6 cm and height 3 cm on its top edge. Find the height of its centre of mass above the bottom edge.

Worked example A3: a framework

A uniform wire is bent into a triangle OAB with $OA = 4a$, $OB = 3a$ and $AB = 5a$, the right angle at O. With OA along the x-axis, find the centre of mass.

Mass $\propto$ length, at each midpoint

$$\bar{x} = \frac{4(2a)+3(0)+5(2a)}{12}$$

$$\bar{y} = \frac{4(0)+3(1.5a)+5(1.5a)}{12}$$

$$(1.5a,\ a)$$

Your turn A3

A uniform wire is bent into three sides of a rectangle: a base of length $6a$ along the x-axis from O, and two upright sides of length $2a$ at its ends. Find the centre of mass.

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STRETCH

Worked example A4: standard results

A uniform lamina is a 6 cm square with a semicircle of radius 3 cm on its top edge. Find the height of its centre of mass above the bottom edge. (Semicircle: G is $\frac{4r}{3\pi}$ above the diameter.)

$$\bar{y} = \frac{36(3) + \frac{9}{2}\pi\left(6 + \frac{4}{\pi}\right)}{36 + \frac{9}{2}\pi}$$

4.20 cm

Your turn A4

A uniform disc has centre O and radius 6 cm. A hole of radius 2 cm is cut out, its centre 3 cm from O. Find the distance of the centre of mass from O

Worked example A5: freely suspended

A uniform rectangle $ABCD$ (mass M, $AB = 8$ cm, $AD = 6$ cm) with a particle of mass M at C hangs from A. Find the angle AB makes with the vertical.

$G = (6,\ 4.5)$ from A; $\tan\theta = \frac{4.5}{6}$

36.9°

Your turn A5

With a particle of mass $2M$ at B instead, the same rectangle hangs from D. Find the angle between DA and the vertical.

Worked example A6: on the point of toppling

A uniform rectangle, 4 cm wide and 10 cm tall, stands on its 4 cm edge on a rough slope at α. Find the greatest α before it topples.

G above the lower corner: $\tan\alpha = \frac{2}{5}$

$\alpha = 21.8°$

Your turn A6

A uniform rectangle, 5 cm wide and 12 cm tall, stands on its 5 cm edge in the same way. Find the greatest α before it topples.

Worked example A7: held by a force

A uniform square $ABCD$ (side $4a$, weight W) with a particle of weight W at C is pivoted at A. Find the vertical force F at B that holds AB horizontal.

$F(4a) = W(2a) + W(4a)$

$F = 1.5W$

Your turn A7

A uniform rectangle $ABCD$ (weight W, $AB = 6a$, $AD = 4a$) with a particle of weight $\frac{1}{2}W$ at B is pivoted at A. Find the vertical force F at C that holds AB horizontal.

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B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Particles of mass 2 kg and 6 kg lie on the x-axis at $x = 0$ and $x = 8$

(a) Find $\bar{x}$

(b) A 4 kg particle is added at $x = 2$. Find $\bar{x}$

(c) Instead the 4 kg particle is placed at $x = p$, and $\bar{x} = 5$. Find p

(d) Instead an m kg particle is placed at $x = 12$, and $\bar{x} = 8$. Find m

[4 marks]

B2 Particles of mass 1 kg, 2 kg and 3 kg are at $(0,0)$, $(4,1)$ and $(2,5)$

(a) Find $\bar{x}$

(b) Find $\bar{y}$

(c) The 3 kg particle moves to $(p,5)$ and $\bar{x} = 3$. Find p

(d) Instead its mass is changed to m kg and $\bar{y} = 3$. Find m

[4 marks]

B3 Find the position of the centre of mass of each uniform body.

(a) A lamina with vertices $(0,0)$, $(6,0)$, $(0,9)$

(b) A semicircular lamina of radius 6 cm: distance from its diameter

(c) A semicircular arc of radius 6 cm: distance from its diameter

(d) A lamina with vertices $(0,0)$, $(6,0)$, $(3,9)$

[4 marks]

B4 A uniform square lamina has corners $(0,0)$, $(6,0)$, $(6,6)$ and $(0,6)$. Find the centre of mass of the lamina when

(a) the square with corners $(4,4)$ and $(6,6)$ is removed

(b) the square with corners $(2,4)$ and $(4,6)$ is removed

(c) instead, that square of side 2 is attached on top, with corners $(2,6)$ and $(4,8)$

(d) the square with corners $(2,2)$ and $(4,4)$ is removed

[4 marks]

B5 Each framework is made of uniform rods. Find the height of its centre of mass above its lowest rod.

(a) An equilateral triangle of three identical rods of length 6 cm.

(b) The same triangle when the base rod has twice the mass per unit length of the other two.

(c) A square of side 4 cm with its top rod removed (three identical rods). **[3 marks]**

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B6 A uniform rectangular lamina $ABCD$ of mass M has $AB = 8$ cm and $AD = 6$ cm.

(a) It is suspended from A. Find the angle between AB and the vertical.

(b) Find the angle between AD and the vertical.

(c) Describe how it hangs from the midpoint of AB

(d) A particle of mass M is fixed at D and it is suspended from A. Find the angle between AB and the vertical.

[4 marks]

STRETCH

B7 A uniform disc has centre O at the origin and radius 4 cm. Find $\bar{x}$ when

(a) a hole of radius 2 cm is cut with centre $(2, 0)$

(b) a hole of radius 1 cm is cut with centre $(2, 0)$

(c) holes of radius 1 cm are cut with centres $(2, 0)$ and $(0, 2)$: find $\bar{x}$ and $\bar{y}$

(d) a hole of radius 2 cm is cut with centre $(2, 0)$ and a hole of radius 1 cm with centre $(-2.5, 0)$

[4 marks]

B8 A uniform wire ABC of mass M is bent at B so that $AB = 6$ cm, $BC = 8$ cm and angle $ABC = 90°$

(a) Find the distance of its centre of mass from AB

(b) Find the distance of its centre of mass from BC

(c) It is suspended from A. Find the angle between AB and the vertical.

(d) A particle of mass kM is fixed at A, and it is suspended from B. BA and BC make equal angles with the vertical. Find k

[4 marks]

B9 Each uniform lamina stands on a rough slope at α, its base up the slope, and does not slide. Find the greatest α before it topples.

(a) A rectangle with base 6 cm and height 8 cm

(b) A rectangle with base 8 cm and height 6 cm

(c) An isosceles triangle with base 6 cm and height 9 cm

(d) A right-angled triangle with base 6 cm and its 9 cm side upright at the lower end

[4 marks]

B10 A uniform square lamina $ABCD$ of side 4 cm and mass M has a particle of mass kM fixed at B. It is suspended from A

(a) Find the angle between AB and the vertical when $k = 0$

(b) Find the angle when $k = 1$

(c) Find the exact value of k for which the angle is $30°$

(d) Describe what happens to the angle as k becomes very large.

[4 marks]

C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 Amir says the centre of mass of a uniform semicircular lamina of radius 6 cm is $\frac{4\pi(6)}{3} = 25.1$ cm from its diameter. Explain how you know he is wrong, and find the correct distance. **[2 marks]**

C2 A uniform wire is bent into an L-shape, $AB = 3$ cm upright and $BC = 5$ cm along the ground. Bea gives the distance of its centre of mass from AB as $\frac{0+2.5}{2} = 1.25$ cm. Explain her error, and find the correct distance. **[2 marks]**

C3 A uniform square lamina $ABCD$ of side 6 cm has a right-angled triangle with shorter sides 3 cm cut from the corner at B. Cal finds the distance of the centre of mass from AD by adding the triangle's moment to the square's, and gets 3.22 cm.

Explain his error, and find the correct distance. **[3 marks]**

C4 A uniform rectangular lamina $ABCD$ has $AB = 10$ cm and $AD = 4$ cm, and is freely suspended from A. Dev uses $\tan\theta = \frac{5}{2}$ and says AB makes 68.2° with the vertical. Explain his error, and find the correct angle.

[3 marks]

STRETCH

C5 Eve says the centre of mass of a uniform trapezium with parallel sides 6 cm and 2 cm, 4 cm apart, is halfway between them, 2 cm from the longer side. Explain her error, and find the correct distance. **[3 marks]**

C6 A lamina is a square of side 4 cm with a square of side 2 cm alongside it, both resting on one line. The small square has three times the mass per unit area of the large one. Fay uses masses 1 : 3 and gets $\bar{x} = 4.25$ cm from the far edge of the large square.

Explain her error, and find $\bar{x}$ **[3 marks]**

C7 A uniform square $ABCD$ of side 4 cm and weight W is pivoted at A and held with AB horizontal by a vertical force F at B. Gus takes moments about A as $W \times AG = F \times 4$, with $AG = 2\sqrt{2}$. Explain his error, and find F **[3 marks]**

C8 A uniform square lamina $OABC$ has side $2a$, with O at the origin, A on the positive x-axis and C on the positive y-axis. A square of side x is removed from the corner at B, where $0 < x < 2a$

(a) Show that the centre of mass of the remaining lamina is at a distance $\frac{8a^3 - 4ax^2 + x^3}{2(4a^2 - x^2)}$ from OC

(3)

(b) Given that the centre of mass lies at the inner corner of the removed square, show that $x = (\sqrt{5} - 1)a$

(2)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 The uniform lamina $ABCED$, shown in Figure 1, is formed from a rectangle $ABCD$, with $AB = 12a$ and $AD = 6a$, and an isosceles triangle BEC attached along BC. The perpendicular distance of E from BC is $6a$

(a) Show that the centre of mass of the lamina is $7.6a$ from AD

(2)

(b) The lamina is freely suspended from A and hangs in equilibrium. Find the angle between AB and the downward vertical.

(1)

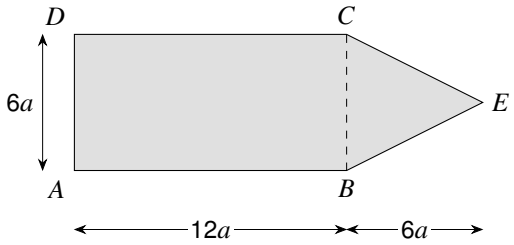


Figure 1

[3 marks]

D2 The uniform rectangular lamina $ABCD$, shown in Figure 2, has $AB = 10a$ and $AD = 4a$. The points E on AB and F on DC are such that $AE = DF = 6a$. The square $EBCF$ is folded along EF so that it lies flat on top of $AEFD$, forming the folded lamina L

(a) Show that the centre of mass of L is $3.4a$ from AD

(3)

(b) L is freely suspended from D. Find the angle between DA and the downward vertical.

(1)

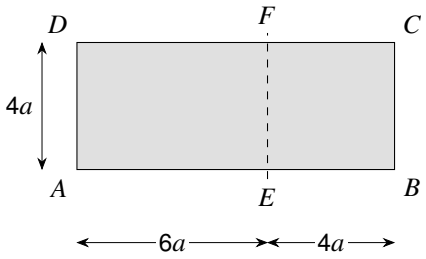


Figure 2

[4 marks]

D3 A uniform wire is bent into the shape shown in Figure 3, a semicircular arc AB of radius r and centre O together with the diameter AB. [The centre of mass of a semicircular arc of radius r is $\frac{2r}{\pi}$ from its centre.]

(a) Show that the centre of mass of the framework is $\frac{2r}{\pi+2}$ from AB

(2)

(b) The framework is freely suspended from A. Find the angle between AB and the downward vertical.

(1)

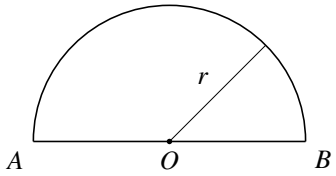


Figure 3

[3 marks]

D4 A uniform circular disc has centre O and radius $3a$. A circular hole of radius a and centre C, where $OC = 2a$, is cut from the disc to form the template shown in Figure 4. The mass of the template is M. The point P is on the rim of the disc, with O, C and P in a straight line.

(a) Find the distance of the centre of mass of the template from O

(2)

(b) A particle of mass kM is fixed at P so that the centre of mass is at O. Find k

(1)

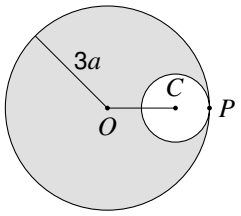


Figure 4

[3 marks]

STRETCH

D5 The lamina in Figure 5 consists of a uniform square $ABCD$ of side $4a$ and a uniform square $BEFG$ of side $2a$, in the same plane, with E on AB produced. The mass per unit area of $BEFG$ is twice the mass per unit area of $ABCD$, and the total mass of the lamina is M

(a) Show that the centre of mass of the lamina is $3a$ from AD

(2)

(b) The lamina is freely pivoted at A and held in equilibrium with AE horizontal by a vertical force of magnitude F acting at E. Find F in terms of M and g

(2)

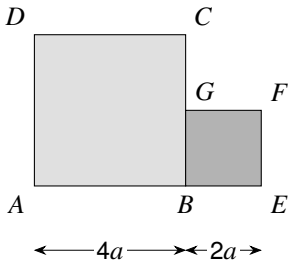


Figure 5

[4 marks]

D6 The template shown in Figure 6 is formed by removing the isosceles triangle AED from a uniform square lamina $ABCD$ of side $6a$. The point E is the centre of the square, so the perpendicular distance of E from AD is $3a$

(a) Show that the centre of mass of the template is $\frac{11a}{3}$ from AD

(2)

(b) The template is freely suspended from B and hangs in equilibrium. Find the angle between BC and the downward vertical.

(2)

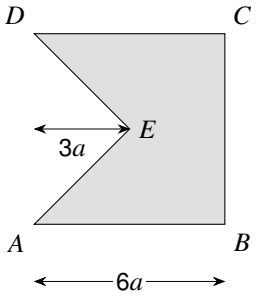


Figure 6

[4 marks]

D7 A uniform triangular lamina ABC has $AB = 6$ cm, $BC = 8$ cm and angle $ABC = 90°$. It stands in a vertical plane with AB along a line of greatest slope of a rough plane inclined at α to the horizontal, as shown in Figure 7. The lamina does not slide.

(a) With B the lower end of AB, find the greatest value of α for which the lamina does not topple.

(2)

(b) The lamina is turned round so that A is the lower end. Find the new greatest value of α

(2)

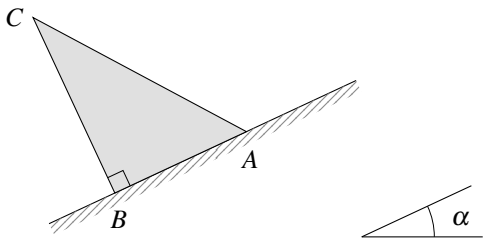


Figure 7

[4 marks]

D8 The template in Figure 8 is made by cutting a circular hole of radius $2a$ from a uniform rectangular lamina $PQRS$, with $PQ = 10a$ and $PS = 6a$. The centre of the hole is $3a$ from PS and $3a$ from PQ

(a) Show that the centre of mass of the template is $\frac{3(25-\pi)}{15-\pi}a$ from PS

(3)

(b) The template is freely suspended from P. Find the angle between PQ and the downward vertical.

(2)

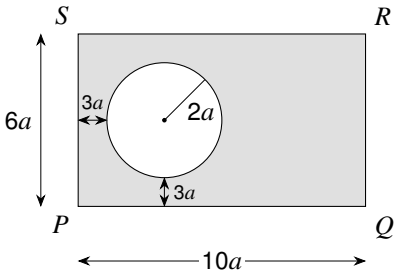


Figure 8

[5 marks]

E Exam practice

Full-size exam questions in the style of WME02. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 A template for a sign is cut from thin uniform card. It is modelled as the uniform lamina $ABCD$, shown in Figure 9, which is in the shape of a trapezium, where

- BC is parallel to AD, with $BC = 4a$ and $AD = 8a$
- AB is perpendicular to both AD and BC, with $AB = 6a$

(a) Show that the centre of mass of the lamina is a distance $\frac{28}{9}a$ from AB

(3)

(b) Find the distance of the centre of mass of the lamina from AD

(2)

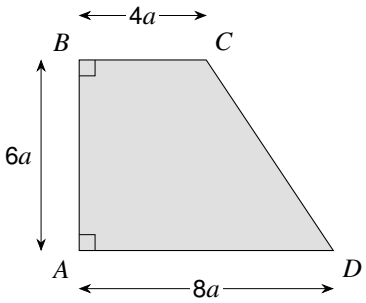


Figure 9

The lamina is freely suspended from B and hangs in equilibrium.

(c) Find the angle between BA and the downward vertical, giving your answer to the nearest degree.

(3)

(Total for Question E1 is 8 marks)

E2 A uniform wire is cut and bent to form the rigid framework shown in Figure 10, which is used as the frame of a small display panel. The framework consists of the rectangle $ABCD$, with $AB = 6a$ and $BC = 4a$, together with a straight piece AM, where M is the midpoint of DC. The framework lies in a vertical plane.

(a) Show that the centre of mass of the framework is a distance $2.7a$ from AD

(3)

(b) Find the distance of the centre of mass of the framework from AB

(2)

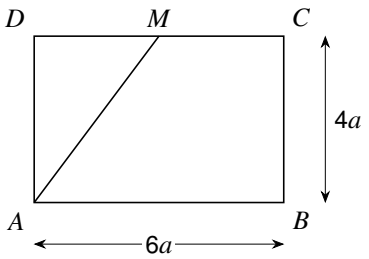


Figure 10

The framework is freely suspended from B and hangs in equilibrium.

(c) Find the angle between BC and the downward vertical.

(3)

The mass of the framework is M. A particle of mass kM is now attached at D. The loaded framework is freely suspended from A and hangs in equilibrium with AB and AD making equal angles with the downward vertical.

(d) Find the value of k

(3)

(Total for Question E2 is 11 marks)

STRETCH

E3 The uniform lamina $ABCED$, shown in Figure 11, consists of a square $ABCD$ of side $6a$ and an isosceles triangle DCE, where $DE = CE$ and the perpendicular distance of E from DC is h

(a) Show that the centre of mass of the lamina is a distance $\dfrac{h^2 + 18ah + 108a^2}{3(h + 12a)}$ from AB

(4)

In the rest of this question $h = 6a$. The lamina stands in a vertical plane with AB along a line of greatest slope of a rough plane inclined at α to the horizontal, as shown in Figure 11. The plane is rough enough to prevent the lamina from sliding down the plane.

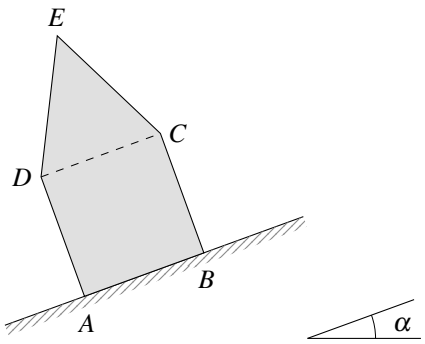


Figure 11

(b) Find the greatest value of α for which the lamina remains in equilibrium.

(3)

The mass of the lamina is M. A particle of mass kM is fixed at E.

(c) Show that the lamina is now on the point of toppling when $\tan\alpha = \dfrac{9(1 + k)}{2(7 + 18k)}$

(3)

(d) Given that the greatest value of α is now $25°$, find the value of k

(2)

(Total for Question E3 is 12 marks)

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