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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 3 (WFM03)

Further Pure 3 Practice Paper 2

Worked Solutions

AIMING FOR A

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM03/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	An identity and a hyperbolic equation	7	2
2	An integral by a hyperbolic substitution	8	3
3	A 3×3 transformation and the image of a line	9	4
4	Two planes and a skew line	9	5
5	Arc length and surface area	10	6
6	A normal to a hyperbola and a locus	10	7
7	An eigenvector with an unknown entry	11	8
8	A reduction formula for $\cosh^n x$	11	9

Question paper & mark scheme | Exam questions by topic | More practice papers

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1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, express $\cosh 2x$ in terms of $\sinh x$

(3)

(b) Hence solve the equation

$$\cosh 2x - 3\sinh x = 3$$

giving your answers as exact logarithms.

(4)

(a) $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$ $\quad \sinh x = \frac{1}{2}(e^{x} - e^{-x})$ M1

$2\sinh^2 x = 2 \times \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = \frac{1}{2}(e^{2x} + e^{-2x}) - 1 = \cosh 2x - 1$ M1

$\boxed{\cosh 2x = 1 + 2\sinh^2 x}$ A1

(b) $1 + 2\sinh^2 x - 3\sinh x = 3 \Rightarrow 2\sinh^2 x - 3\sinh x - 2 = 0$ M1

$(2\sinh x + 1)(\sinh x - 2) = 0 \Rightarrow \sinh x = 2$ or $\sinh x = -\frac{1}{2}$ A1

(keep both: sinh takes every real value)

$x = \text{arsinh}\, s = \ln(s + \sqrt{s^2 + 1})$: $\ln(2 + \sqrt{5})$, $\ln(-\frac{1}{2} + \sqrt{\frac{5}{4}}) = \ln\frac{\sqrt{5} - 1}{2}$ M1

$\boxed{x = \ln(2 + \sqrt{5}),\ x = \ln((\sqrt{5} - 1)/2)}$ A1

(Total for Question 1 is 7 marks)

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2. Use a suitable substitution to find the exact value of

$$\int_0^4 \sqrt{x^2+9}\,\mathrm{d}x$$

giving your answer in the form $a+b\ln 3$, where a and b are rational numbers.

(8)

Let $x = 3\sinh u$, so $dx = 3\cosh u\,du$ — M1

$\sqrt{9\sinh^2 u + 9} = 3\sqrt{\sinh^2 u + 1} = 3\cosh u$: integrand $3\cosh u \times 3\cosh u = 9\cosh^2 u$ — A1

$\cosh^2 u = \frac{1}{2}(1 + \cosh 2u)$: $\int 9\cosh^2 u\,du = \frac{9}{2}\int (1 + \cosh 2u)\,du$ — M1

$= \frac{9}{2}[u + \frac{1}{2}\sinh 2u] = \frac{9}{2}[u + \sinh u \cosh u]$ — A1

Limits: $x = 0 \rightarrow u = 0$; $x = 4 \rightarrow \sinh u = \frac{4}{3}$, $u = \operatorname{arsinh}\frac{4}{3} = \ln(\frac{4}{3} + \frac{5}{3}) = \ln 3$ — M1

At $u = \ln 3$: $\sinh u = \frac{4}{3}$, $\cosh u = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$ — A1

$\frac{9}{2}(\ln 3 + \frac{4}{3} \times \frac{5}{3}) - 0 = \frac{9}{2}(\ln 3 + \frac{20}{9})$ — M1

$10 + (9/2)\ln 3$ — A1

(Total for Question 2 is 8 marks)

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3.

$$\mathbf{M} = \begin{pmatrix} k & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & k \end{pmatrix}$$

where k is a constant.

(a) Find $\det \mathbf{M}$ in terms of k, and explain why $\mathbf{M}$ is non-singular for every real value of k

(3)

A solid of volume 2 cm^3 is transformed by $\mathbf{M}$ into a solid of volume 38 cm^3

(b) Given that $k > 0$, find the value of k

(2)

Using this value of k, the line l with equation

$$\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{1}$$

is transformed by $\mathbf{M}$ into the line l'

(c) Find a Cartesian equation of l'

(4)

(a) Expand along row 1: $k(2k - 0) - 1(0 - 1) + 0 = 2k^2 + 1$ M1

$\det M = 2k^2 + 1$ A1

$k^2 \geqslant 0$, so $2k^2 + 1 \geqslant 1 > 0$: det M is never 0, so M is non-singular for all k B1

(b) Volume scale factor = $|\det M| = \frac{38}{2} = 19$: $2k^2 + 1 = 19 \Rightarrow k^2 = 9$ M1

$k > 0$: $k = 3$ A1

(c) l: $\mathbf{r} = (1, 0, -2) + t(2, -1, 1)$. Transform the point and the direction:

$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$ M1 $\quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix}$ M1

$(x - 3)/5 = (y + 2)/(-1) = (z + 5)/5$ A1

(Total for Question 3 is 9 marks)

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4. The planes Π_1 and Π_2 have equations

$$\Pi_1 : x + y - z = 1 \qquad \Pi_2 : 2x - y + z = 5$$

(a) Find a vector equation of the line L in which Π_1 and Π_2 meet.

(5)

The line m has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{k})$

(b) Find the shortest distance between L and m, giving your answer as an exact surd.

(4)

(a) Direction of $L = n_1 \times n_2 = (1, 1, -1) \times (2, -1, 1)$ M1

$= (1 \times 1 - (-1)(-1),\ (-1)(2) - 1 \times 1,\ 1(-1) - 1 \times 2) = (0, -3, -3)$, or $(0, 1, 1)$ A1

Point on both planes: add the equations: $3x = 6$, $x = 2$; then $y - z = -1$, take $z = 1$, $y = 0$ M1

$(2, 0, 1)$: check $2 + 0 - 1 = 1$ ✓, $4 - 0 + 1 = 5$ ✓ A1

$\boxed{r = 2i + k + \lambda(j + k)}$ A1

(b) Common perpendicular: $(0, 1, 1) \times (1, 0, 2)$

$= (1 \times 2 - 1 \times 0,\ 1 \times 1 - 0 \times 2,\ 0 \times 0 - 1 \times 1) = (2, 1, -1)$ M1

Vector from L to m: $(1, 2, 0) - (2, 0, 1) = (-1, 2, -1)$ M1

Distance $= \dfrac{|(-1, 2, -1)\cdot(2, 1, -1)|}{|(2, 1, -1)|} = \dfrac{|-2 + 2 + 1|}{\sqrt{6}} = \dfrac{1}{\sqrt{6}}$ M1

$\boxed{\sqrt{6}/6}$ A1

(Total for Question 4 is 9 marks)

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5. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The curve C has parametric equations

$$x = 2t^3, \qquad y = 3t^2, \qquad 0 \leqslant t \leqslant 1$$

(a) Find the exact length of C

(4)

(b) The curve C is rotated through 2π radians about the x-axis. Find the exact area of the surface generated.

(6)

(a) $\frac{dx}{dt} = 6t^2$, $\frac{dy}{dt} = 6t$ M1

$\sqrt{(6t^2)^2 + (6t)^2} = \sqrt{36t^2(t^2 + 1)} = 6t\sqrt{1 + t^2}$ $(t \geqslant 0)$ M1

Length $= \int_0^1 6t\sqrt{1 + t^2}\, dt = [2(1 + t^2)^{3/2}]_0^1$ A1

$= 2 \times 2\sqrt{2} - 2 = \boxed{4\sqrt{2} - 2}$ A1

(b) $S = 2\pi \int y\, ds = 2\pi \int_0^1 3t^2 \times 6t\sqrt{1 + t^2}\, dt$ M1

$= 36\pi \int_0^1 t^3\sqrt{1 + t^2}\, dt$ A1

$u = 1 + t^2$, $du = 2t\, dt$, $t^2 = u - 1$; $t = 0 \to u = 1$, $t = 1 \to u = 2$ M1

$= 36\pi \int_1^2 (u - 1)\sqrt{u} \times \frac{1}{2}\, du = 18\pi \int_1^2 (u^{3/2} - u^{1/2})\, du$ A1

$= 18\pi[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}]_1^2 = 18\pi(\frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \frac{2}{5} + \frac{2}{3}) = 18\pi(\frac{4\sqrt{2} + 4}{15})$ M1

$\boxed{(24\pi/5)(\sqrt{2} + 1)}$ (no end discs: only the curved surface) A1

(Total for Question 5 is 10 marks)

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6.

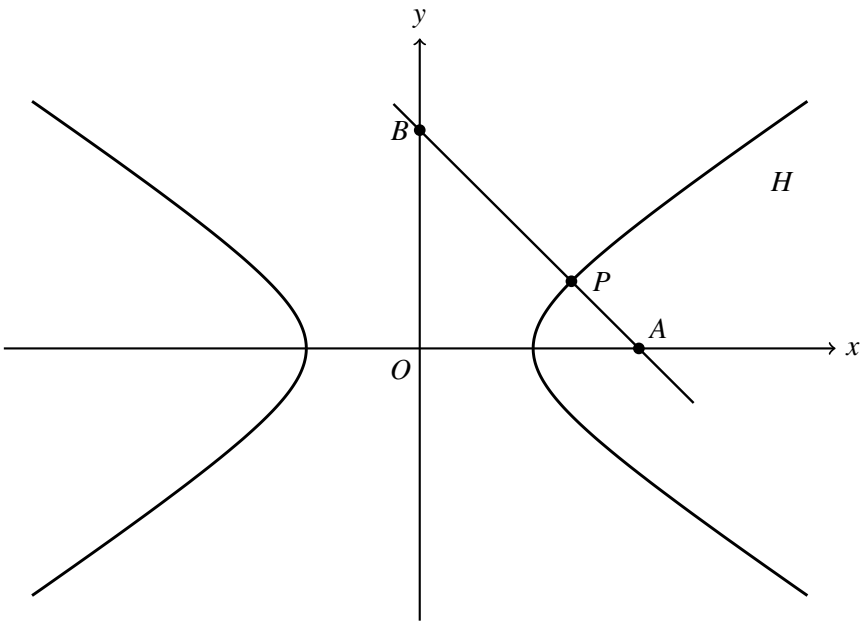

Figure 1

Figure 1 shows the hyperbola H with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$. The point $P(3\cosh t, 2\sinh t)$, $t > 0$, lies on H. The normal to H at P meets the x-axis at A and the y-axis at B

(a) Find an equation of the normal to H at P, giving your answer in the form $px\sinh t + qy\cosh t = r\sinh t\cosh t$, where p, q and r are integers.

(5)

(b) Find the coordinates of A and B

(2)

(c) Find a Cartesian equation of the locus of the midpoint of AB as t varies, stating any restrictions on x and y

(3)

(a) $\frac{dx}{dt} = 3\sinh t$, $\frac{dy}{dt} = 2\cosh t \Rightarrow \frac{dy}{dx} = \frac{2\cosh t}{3\sinh t}$ M1 A1

Normal gradient $-\frac{3\sinh t}{2\cosh t}$: $y - 2\sinh t = -\frac{3\sinh t}{2\cosh t}(x - 3\cosh t)$ M1 M1

× 2 cosh t and rearrange: $3x\sinh t + 2y\cosh t = 13\sinh t\cosh t$ A1

(b) $y = 0$: $A((13/3)\cosh t, 0)$ $x = 0$: $B(0, (13/2)\sinh t)$ B1 B1

(c) Midpoint: $x = \frac{13}{6}\cosh t$, $y = \frac{13}{4}\sinh t$ M1

$\cosh^2 t - \sinh^2 t = 1$: $(\frac{6x}{13})^2 - (\frac{4y}{13})^2 = 1$ M1

$36x^2 - 16y^2 = 169$, with $x > 13/6$ and $y > 0$ ($t > 0$ gives $\cosh t > 1$, $\sinh t > 0$) A1

(Total for Question 6 is 10 marks)

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7.

$$\mathbf{A} = \begin{pmatrix} 3 & a & 1 \\ a & 2 & a \\ 1 & a & 3 \end{pmatrix}$$

where a is a constant. Given that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$

(a) find the value of a and the corresponding eigenvalue

(3)

(b) find the other two eigenvalues of $\mathbf{A}$

(4)

(c) find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$

(4)

(a) $A(1, 1, 1) = (3 + a + 1,\ a + 2 + a,\ 1 + a + 3) = (4 + a,\ 2 + 2a,\ 4 + a)$ M1

An eigenvector: all components equal λ: $4 + a = 2 + 2a$ M1

$a = 2$, eigenvalue 6 A1

(b) $\det(A - \lambda I)$ with $a = 2$:

$(3 - \lambda)[(2 - \lambda)(3 - \lambda) - 4] - 2[2(3 - \lambda) - 2] + [4 - (2 - \lambda)]$ M1

$= -\lambda^3 + 8\lambda^2 - 12\lambda$ A1

$\lambda = 6$ is a root: $-\lambda(\lambda^2 - 8\lambda + 12) = -\lambda(\lambda - 6)(\lambda - 2)$ M1

$\lambda = 2,\ \lambda = 0$ A1

(c) $\lambda = 2$: $x + 2y + z = 0,\ 2x + 2z = 0 \Rightarrow (1, 0, -1)$; $\lambda = 0$: $(1, -2, 1)$ M1 A1

Normalise: $(1, 1, 1)/\sqrt{3},\ (1, 0, -1)/\sqrt{2},\ (1, -2, 1)/\sqrt{6}$ (they are already orthogonal) M1

$$P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \qquad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ A1

(The columns of P are in the same order as the eigenvalues in D.)

(Total for Question 7 is 11 marks)

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8.

$$I_n = \int_0^{\ln 2} \cosh^n x \, dx, \qquad n \geqslant 0$$

(a) Show that, for $n \geqslant 2$,

$$nI_n = \frac{3}{4}\left(\frac{5}{4}\right)^{n-1} + (n-1)I_{n-2}$$

(6)

(b) Hence find the exact value of I_4

(5)

(a) $I_n = \int \cosh^{n-1} x \times \cosh x \, dx$: parts with $u = \cosh^{n-1} x$, $dv/dx = \cosh x$ M1

$= [\sinh x \cosh^{n-1} x] - \int \sinh x \times (n-1)\cosh^{n-2} x \sinh x \, dx$ M1

$= [\sinh x \cosh^{n-1} x] - (n-1)\int \sinh^2 x \cosh^{n-2} x \, dx$ A1

$\sinh^2 x = \cosh^2 x - 1$: $I_n = [\sinh x \cosh^{n-1} x] - (n-1)(I_n - I_{n-2})$ M1

$x = \ln 2$: $\sinh = \frac{2 - 1/2}{2} = \frac{3}{4}$, $\cosh = \frac{2 + 1/2}{2} = \frac{5}{4}$; $x = 0$: $\sinh = 0$ M1

$nI_n = \frac{3}{4}(\frac{5}{4})^{n-1} + (n-1)I_{n-2}$ as required A1

(b) $I_0 = \int_0^{\ln 2} 1 \, dx = \ln 2$ B1

$n = 2$: $2I_2 = \frac{3}{4} \times \frac{5}{4} + I_0 = \frac{15}{16} + \ln 2$ M1

$I_2 = \frac{15}{32} + \frac{1}{2}\ln 2$ A1

$n = 4$: $4I_4 = \frac{3}{4} \times \frac{125}{64} + 3I_2 = \frac{375}{256} + \frac{360}{256} + \frac{3}{2}\ln 2 = \frac{735}{256} + \frac{3}{2}\ln 2$ M1

$I_4 = 735/1024 + (3/8)\ln 2$ A1

(Total for Question 8 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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