



eClassroom

A-LEVEL MATHEMATICS • PURE 4

Integration

Worked Solutions

Pearson Edexcel International A Level (WMA14) & A Level (9MA0) • Calculator allowed: show every step

Full worked solutions for the free Integration worksheet, A-Level and International A-Level Maths



Tap a section to jump straight to it

Full worked solutions for every question on the Pure 4 Integration worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet

Exam questions by topic

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A

Your turn answers

Compare each line of your working with the solution.

CORE

A1 $dx = du$, limits 1 to 4: $\int_1^4 (u-1)u^{\frac{1}{2}} du = \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right]_1^4$

$$\left(\frac{64}{5} - \frac{16}{3} \right) - \left(\frac{2}{5} - \frac{2}{3} \right): \boxed{\frac{116}{15}}$$

A2 $u = \ln x, \frac{dv}{dx} = x: \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_1^e = \left(\frac{e^2}{2} - \frac{e^2}{4} \right) - \left(0 - \frac{1}{4} \right): \boxed{\frac{e^2 + 1}{4}}$

A3 $\frac{3}{(x+1)(2-x)} = \frac{1}{x+1} + \frac{1}{2-x}; \int \frac{1}{2-x} dx = -\ln(2-x)$

$$[\ln(x+1) - \ln(2-x)]_0^1 = (\ln 2 - 0) - (0 - \ln 2): \boxed{\ln 4}$$

STRETCH

A4 $y^2 = 1 + 2e^{-x} + e^{-2x}: \pi \left[x - 2e^{-x} - \frac{1}{2}e^{-2x} \right]_0^{\ln 2}$

$$\pi \left(\ln 2 - 1 - \frac{1}{8} \right) - \pi \left(-2 - \frac{1}{2} \right): \boxed{\pi \left(\ln 2 + \frac{11}{8} \right)}$$

A5 $\int \frac{1}{y+1} dy = \int 2x dx: \ln(y+1) = x^2 + c, c = \ln 2$

$$\boxed{y = 2e^{x^2} - 1}$$

A6 Parts twice: $[x^2 \sin x + 2x \cos x - 2 \sin x]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} + 0 - 2: \boxed{\frac{\pi^2}{4} - 2}$

A7 $\pi \int_0^{\frac{\pi}{2}} \cos^2 t \times 2 \cos t dt = 2\pi \int_0^{\frac{\pi}{2}} \cos t (1 - \sin^2 t) dt = 2\pi \left[\sin t - \frac{\sin^3 t}{3} \right]_0^{\frac{\pi}{2}}: \boxed{\frac{4\pi}{3}}$



B

Fluency solutions

One mark for each correct part (each needs +c). The notes in italics explain why each sequence was chosen.

CORE

B1 (a) $\frac{(x^2-3)^6}{6}$ (b) $e^{\sin x}$ (c) $\frac{1}{2} \ln(x^2+4)$ (d) $\frac{2}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}}$

Replace dx using $\frac{du}{dx}$; in (d) write $x = u + 2$ first.

B2 (a) $\frac{15}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $u = x + 1: \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_1^4: \frac{8}{3}$

Change the limits to u values, then there is no need to go back to x

B3 (a) $e^{2x} \left(\frac{x}{2} - \frac{1}{4} \right)$ (b) $\frac{x \sin 3x}{3} + \frac{\cos 3x}{9}$ (c) $x \ln x - x$ (d) $\frac{x^3}{3} \ln x - \frac{x^3}{9}$

Let $u = x$ unless there is a $\ln x$; then $u = \ln x$ (in (c), $\frac{dv}{dx} = 1$).

B4 (a) $\ln|x-1| - \ln|x+1|$ (b) $\ln|x| - \ln|x+3|$
 (c) $\frac{2}{2x+1} - \frac{1}{x+1}: \ln|2x+1| - \ln|x+1|$ (d) $x + \ln|x+1|$

In (c) the 2 cancels the $\frac{1}{2}$ from $2x+1$; in (d) divide first.

CORE

B5 (a) $\pi \int_0^4 x dx: 8\pi$ (b) $\pi \int_0^1 x^4 dx: \frac{\pi}{5}$ (c) $\pi \int_1^3 x^{-2} dx: \frac{2\pi}{3}$

Square y before integrating; keep π outside.

B6 (a) $y = 2e^{3x}$ (b) $\frac{y^2}{2} = \frac{x^2}{2} + c: y^2 = x^2 - 7$ (c) $\ln y = \sin x + 1: y = e^{1+\sin x}$ (d) $e^y = x^2 + 1: y = \ln(x^2 + 1)$

Separate, integrate, add c , use the condition, then rearrange.



STRETCH

B7 (a) $\int \frac{u-1}{u} du$: $e^x + 1 - \ln(e^x + 1) + c$ (b) $[\ln(1 + \sin x)]_0^{\frac{\pi}{2}}$: $\ln 2$

(c) $dx = 2u du$: $\int \frac{2}{u+1} du$: $2\ln(1 + \sqrt{x}) + c$ (d) $\left[\frac{2}{9}(x^3 + 1)^{\frac{3}{2}}\right]_0^2$: $\frac{52}{9}$

In (a) and (c) x must be fully replaced; (b) and (d) are reverse chain rule.

B8 (a) $-e^{-x}(x^2 + 2x + 2)$ (b) $\frac{x(x+1)^6}{6} - \frac{(x+1)^7}{42}$

(c) $-\frac{\ln x}{x} - \frac{1}{x}$ (d) $\frac{1}{2} \int x \sin 2x dx$: $-\frac{x \cos 2x}{4} + \frac{\sin 2x}{8}$

Parts twice, a bracket as $\frac{dv}{dx}$, $\ln x$ as u , and a double angle before parts.

B9 (a) $\pi[\tan x]$: $\sqrt{3}\pi$ (b) $\pi \left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2} \right]_0^1$: $\pi \left(\frac{e^2}{2} + 2 - \frac{e^{-2}}{2} \right)$

(c) $\frac{\pi}{2} \int_0^\pi (1 - \cos 2x) dx$: $\frac{\pi^2}{2}$ (d) $\pi \int_{-1}^1 (4 - (1+x^2)^2) dx$: $\frac{64\pi}{15}$

In (d) subtract the inner volume from the outer one, after squaring each.

B10 (a) $\ln(y-2) = \ln x + \ln 3$: $y = 3x + 2$ (b) $\sin y = -\cos x$

(c) $\frac{y^2}{2} = e^x(x-1) + \frac{3}{2}$: $y^2 = 2e^x(x-1) + 3$ (d) $\ln y = \ln x - \ln(x+2) + \ln 2$: $y = \frac{2x}{x+2}$

The x side may need parts (c) or partial fractions (d) before the constant is found.

C

Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 She left x in and did not replace dx : $du = 2x dx$, so $x dx = \frac{1}{2} du$

$$\int \frac{1}{2} u^3 du: \frac{(x^2 + 1)^4}{8} + c$$

C2 $\int \frac{1}{2x-1} dx = \frac{1}{2} \ln |2x-1|$: he lost the $\frac{1}{2}$: $2\ln |2x-1| + c$



C3 He squared each term separately; $(1 + \sqrt{x})^2 = 1 + 2\sqrt{x} + x$

$$\pi \left[x + \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2} \right]_0^4 = \pi \left(4 + \frac{32}{3} + 8 \right): \quad \boxed{\frac{68\pi}{3}}$$

C4 She left out the constant: $e^{2x} = 1$ at $x = 0$, not 3. With $\ln y = 2x + c$, $c = \ln 3$: $\boxed{y = 3e^{2x}}$

STRETCH

C5 She kept the x limits. With $u = x^2$ the limits are $u = 0$ and $u = 4$: $\frac{1}{2}[e^u]_0^4$: $\boxed{\frac{1}{2}(e^4 - 1)}$

C6 $\frac{dv}{dx} = \ln x$ needs $\int \ln x dx$ first, which itself needs parts. Use $u = \ln x$, $\frac{dv}{dx} = x$

$$\boxed{\frac{x^2}{2} \ln x - \frac{x^2}{4} + c}$$

C7 She left out $\frac{dx}{dt} = 2t$: $V = \pi \int y^2 \frac{dx}{dt} dt = \pi \int_0^1 8t^3 dt$: $\boxed{2\pi}$

C8 (a) $u = \ln x$, $\frac{dv}{dx} = 1$: $x \ln x - \int x \times \frac{1}{x} dx$: $\boxed{x \ln x - x + c}$

(b) $u = (\ln x)^2$, $\frac{dv}{dx} = 1$: $x(\ln x)^2 - \int 2 \ln x dx = x(\ln x)^2 - 2x \ln x + 2x$

$$[e - 2e + 2e] - [0 - 0 + 2]: \quad \boxed{e - 2}$$

D

Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 $u = x$, $\frac{dv}{dx} = \sin 3x$: $\left[-\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} \right]_0^{\frac{\pi}{3}} = -\frac{\pi}{9} \times (-1) + 0$: $\boxed{\frac{\pi}{9}}$

D2 (a) $\boxed{\frac{3}{x-1} + \frac{2}{x+2}}$

(b) $[3 \ln(x-1) + 2 \ln(x+2)]_2^4 = 3 \ln 3 + 2 \ln 6 - 2 \ln 4 = \ln \frac{27 \times 36}{16}$: $\boxed{k = \frac{243}{4}}$



D3 $\int y \, dy = \int 6x^2 \, dx$: $\frac{y^2}{2} = 2x^3 + c$; $\frac{9}{2} = 2 + c$, $c = \frac{5}{2}$: $y^2 = 4x^3 + 5$

D4 $y^2 = x + 2 + \frac{1}{x}$: $\pi \left[\frac{x^2}{2} + 2x + \ln x \right]_1^4 = \pi \left(16 + \ln 4 - \frac{5}{2} \right)$: $\pi \left(\frac{27}{2} + 2 \ln 2 \right)$

STRETCH

D5 $x = u^2$, $dx = 2u \, du$, limits 1 and 2: $\int_1^2 2ue^u \, du = [2e^u(u-1)]_1^2$: $2e^2$

D6 (a) Area = $\int y \frac{dx}{dt} \, dt$ with $\frac{dx}{dt} = 2t$, and t runs from 0 to π : $\int_0^\pi 2t \sin t \, dt$

(b) $[2 \sin t - 2t \cos t]_0^\pi = 0 + 2\pi$: 2π

D7 (a) $\int \frac{1}{\theta - 20} \, d\theta = \int -k \, dt$: $\ln(\theta - 20) = -kt + c$, $c = \ln 70$: $\theta = 20 + 70e^{-kt}$

(b) $35 = 70e^{-10k}$: $k = \frac{\ln 2}{10}$; $10 = 70e^{-kt}$, $t = \frac{\ln 7}{k} = \frac{10 \ln 7}{\ln 2}$: **28.1 minutes**

D8 (a) $V = \pi \int y^2 \, dx$, $y^2 = x^2 e^x$: $\pi \int_0^1 x^2 e^x \, dx$

(b) Parts twice: $\int x^2 e^x \, dx = e^x (x^2 - 2x + 2)$; $\pi [e^x (x^2 - 2x + 2)]_0^1 = \pi(e - 2)$: $\pi(e - 2)$

E

Exam practice solutions

Mark-scheme notes are shown on the right of each line.



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CORE

Exam questions for everyone

E1 $x = \frac{u-1}{2}$, $dx = \frac{1}{2}du$, limits $u = 1$ and $u = 9$

B1

$$\int_1^9 \frac{u-1}{2u^2} \times \frac{1}{2} du = \frac{1}{4} \int_1^9 \left(\frac{1}{u} - \frac{1}{u^2} \right) du$$

M1

$$\frac{1}{4} \left[\ln u + \frac{1}{u} \right]_1^9$$

dM1 A1

$$\frac{1}{4} \left(\ln 9 + \frac{1}{9} - 1 \right) = \frac{1}{2} \ln 3 - \frac{2}{9}: \quad a = \frac{1}{2}, b = -\frac{2}{9}$$

A1

E2 (a) $9 \equiv A(x+2) + B(x+1)$: $A = 9$, $B = -9$:

$$\frac{9}{x+1} - \frac{9}{x+2}$$

M1 A1

(b) $V = \pi \int_0^2 y^2 dx = \pi \int_0^2 \frac{9}{(x+1)(x+2)} dx$

B1

$$9\pi [\ln(x+1) - \ln(x+2)]_0^2 = 9\pi(\ln 3 - \ln 4 + \ln 2)$$

M1 A1

$$9\pi \ln \frac{3}{2}$$

A1

STRETCH

E3 (a) $\frac{1}{2-x} - \frac{1}{3-x}$

M1 A1

(b) $\int \left(\frac{1}{2-x} - \frac{1}{3-x} \right) dx = \int k dt$: $\ln(3-x) - \ln(2-x) = kt + c$

M1 A1

$t = 0, x = 0$: $c = \ln \frac{3}{2}$, so $kt = \ln \left(\frac{2(3-x)}{3(2-x)} \right)$

A1*

(c) $5k = \ln \frac{2 \times 2}{3 \times 1}$: $k = \frac{1}{5} \ln \frac{4}{3}$

B1

(d) $e^{kt} = \frac{2(3-x)}{3(2-x)}$, so $x = \frac{6(e^{kt} - 1)}{3e^{kt} - 2}$

M1

as $t \rightarrow \infty$, $x \rightarrow \frac{6}{3}$: **2 grams in the long term**

A1

